STAT/MA 41600
In-Class Problem Set #7: September 9, 2015
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Problem Set 7 Answers

1. We have $X \le x$ if and only if all of the values on the three dice are less than or equal to x. Thus, $P(X \le x) = x^3/216$. So we get:
$P(X = 1) = P(X \le 1) = 1/216$
$P(X = 2) = P(X \le 2) - P(X \le 1) = 8/216 - 1/216 = 7/216$
$P(X = 3) = P(X \le 3) - P(X \le 2) = 27/216 - 8/216 = 19/216$
$P(X = 4) = P(X \le 4) - P(X \le 3) = 64/216 - 27/216 = 37/216$
$P(X = 5) = P(X \le 5) - P(X \le 4) = 125/216 - 64/216 = 61/216$
$P(X = 6) = P(X \le 6) - P(X \le 5) = 216/216 - 125/216 = 91/216$
By the way, these probabilities (of course) sum to 1.

2. The probabilities are:

$$P(X = 0) = \frac{\binom{3}{0}\binom{6}{3}}{\binom{9}{3}} = \frac{5}{21}; \qquad P(X = 1) = \frac{\binom{3}{1}\binom{6}{2}}{\binom{9}{3}} = \frac{15}{28};$$
$$P(X = 2) = \frac{\binom{3}{2}\binom{6}{1}}{\binom{9}{3}} = \frac{3}{14}; \qquad P(X = 3) = \frac{\binom{3}{3}\binom{6}{0}}{\binom{9}{3}} = \frac{1}{84}.$$

The general formula is $P(X = x) = \binom{3}{x}\binom{6}{3-x}/\binom{9}{3}$. Again, the probabilities sum to 1.

3a. We have $X > x$ if the first x rolls have no 3's. Thus, we have $P(X > x) = (5/6)^x$.
3b. From **(3a)**, we compute

$$P(X = x) = P(X > x-1) - P(X > x) = (5/6)^{x-1} - (5/6)^x = (1-5/6)(5/6)^{x-1} = (1/6)(5/6)^{x-1}.$$

3c. We can verify

$$\sum_{x=1}^{\infty}(1/6)(5/6)^{x-1} = (1/6)\sum_{x=1}^{\infty}(5/6)^{x-1} = (1/6)(1+5/6+(5/6)^2+(5/6)^3+\cdots) = (1/6)\frac{1}{1-5/6} = 1.$$

4. We see that $X = x$ if the xth marble is red and any other afterwards (from the $(x+1)$st marble to the 8th marble) is red too. There are $\binom{8}{2} = 28$ ways to choose which two marbles are red. So the desired probability is $P(X = x) = (8 - x)/28$.

If you did not notice the fact above, you can also go case by case, to compute:
$P(X = 1) = 2/8 = 1/4 = 7/28$
$P(X = 2) = (6/8)(2/7) = 3/14 = 6/28$
$P(X = 3) = (6/8)(5/7)(2/6) = 5/28$
$P(X = 4) = (6/8)(5/7)(4/6)(2/5) = 1/7 = 4/28$
$P(X = 5) = (6/8)(5/7)(4/6)(3/5)(2/4) = 3/28$
$P(X = 6) = (6/8)(5/7)(4/6)(3/5)(2/4)(2/3) = 1/14 = 2/28$
$P(X = 7) = (6/8)(5/7)(4/6)(3/5)(2/4)(1/3)(2/2) = 1/28$
Again, the probabilities do sum to 1.