STAT/MA 41600
In-Class Problem Set #9: September 14, 2015
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**Problem Set 9 Answers**

**1.** The probability mass function of $X$ is $p_X(x) = 1/6$ for integers $1 \leq x \leq 6$. The probability mass function of $Y$ is $p_Y(2) = 1/16$, $p_Y(3) = 2/16$, $p_Y(4) = 3/16$, $p_Y(5) = 4/16$, $p_Y(6) = 3/16$, $p_Y(7) = 2/16$, $p_Y(8) = 1/16$. Also, $X$ and $Y$ are independent in this problem. So the desired probability is

$$P(X < 2, Y = 2) + P(X < 3, Y = 3) + P(X < 4, Y = 4) + P(X < 5, Y = 5) + P(X < 6, Y = 6) + P(Y = 7) + P(Y = 8)$$

which simply turns out to be:

$$\left(\frac{1}{6}\right)\left(\frac{1}{16}\right)+\left(\frac{2}{6}\right)\left(\frac{2}{16}\right)+\left(\frac{3}{6}\right)\left(\frac{3}{16}\right)+\left(\frac{4}{6}\right)\left(\frac{4}{16}\right)+\left(\frac{5}{6}\right)\left(\frac{3}{16}\right)+\frac{2}{16}+\frac{1}{16} = 63/96 = 21/32 = 0.65625.$$

**2a.** We have $p_{X|Y}(0 \mid 0) = (4/6)(3/5) = 2/5$, $p_{X|Y}(1 \mid 0) = (4/6)(2/5) + (2/6)(4/5) = 8/15$, and $p_{X|Y}(2 \mid 0) = (2/6)(1/5) = 1/15$. We verify $2/5 + 8/15 + 1/15 = 1$.
**2b.** We have $p_{X|Y}(0 \mid 1) = (5/6)(4/5) = 2/3$, $p_{X|Y}(1 \mid 1) = (5/6)(1/5) + (1/6)(5/5) = 1/3$. We verify $2/3 + 1/3 = 1$.

**3a.** We have

- $p_{X,Y}(0, 0) = 3/5$ ($X = Y = 0$ exactly when the last fish is red);
- $p_{X,Y}(0, 1) = (1/5)(3/4) = 3/20$ ($X = 0$ and $Y = 1$ if the last is blue & the 4th is red);
- $p_{X,Y}(1, 0) = (1/5)(3/4) = 3/20$ ($X = 1$ and $Y = 0$ if the last is green & the 4th is red);
- $p_{X,Y}(1, 1) = (2/5)(1/4) = 1/10$ ($X = Y = 1$ if the last two fish are green and blue).

**3b.** We verify that $3/5 + 3/20 + 3/20 + 1/10 = 1$.
**3c.** We have $X = 1$ if the green fish is last, or if the green fish is 4th and the blue fish is last. So $p_X(1) = 1/5 + (1/5)(1/4) = 1/4$.

Another way to see this is that, when paying attention to only the 3 reds and the 1 green, we have $X = 1$ only if the green comes after all 3 reds, so $p_X(1) = 1/4$.

Similarly, we have $p_Y(1) = 1/4$.
**3d.** The random variables $X$ and $Y$ are dependent since $p_{X,Y}(1, 1) \neq p_X(1)p_Y(1)$.

**4.** If we are given $Y = y$, then the first $y - 1$ rolls do not have any occurrences of 4, but the other 5 results are equally likely. So the probability that exactly $x$ out of these $y - 1$ results are 3's is: $p_{X|Y}(x \mid y) = \binom{y-1}{x}(1/5)^x(4/5)^{y-1-x}$.