STAT/MA 41600
In-Class Problem Set #12: September 21, 2015
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Problem Set 12 Answers

1a. The mass of X is $P(X=j)=\binom{3}{j}(.60)^j(.40)^{3-j}$ for $0\leq j\leq 3$, so we get $\mathbb{E}(X^2)=0^2P(X=0)+1^2P(X=1)+2^2P(X=2)+3^2P(X=3)=99/25=3.96$.
1b. Let X_1,X_2,X_3 denote (respectively) whether the 1st, 2nd, 3rd person is a fan of da Bears. Then $X=X_1+X_2+X_3$, so $\mathbb{E}(X^2)=\mathbb{E}((X_1+X_2+X_3)^2)$, which has 6 terms of the form $\mathbb{E}(X_iX_j)$ (for $i\neq j$) and 3 terms of the form $\mathbb{E}(X_j^2)$. Since X_i and X_j are independent for $i\neq j$, then $\mathbb{E}(X_iX_j)=\mathbb{E}(X_i)\mathbb{E}(X_j)=(.6)(.6)=.36$. Also, since indicators only take on values 0 or 1, then $\mathbb{E}(X_j^2)=\mathbb{E}(X_j)=.6$. Thus $\mathbb{E}(X^2)=(6)(.36)+(3)(.6)=99/25=3.96$.
1c. We have $\mathrm{Var}(X)=\mathbb{E}(X^2)-(\mathbb{E}(X))^2=3.96-(1.8)^2=0.72$.
1d. Since the X_j's are independent, $\mathrm{Var}(X)=\mathrm{Var}(X_1+X_2+X_3)=\mathrm{Var}(X_1)+\mathrm{Var}(X_2)+\mathrm{Var}(X_3)$. We have $\mathrm{Var}(X_j)=\mathbb{E}(X_j^2)-(\mathbb{E}(X_j))^2=.6-(.6)^2=0.24$, so $\mathrm{Var}(X)=3(0.24)=0.72$.

2a. The mass of Y is $P(Y=j)=\binom{2}{j}\binom{6}{2-j}/\binom{8}{2}$ for $0\leq j\leq 2$, so we get $\mathbb{E}(Y^2)=0^2P(Y=0)+1^2P(Y=1)+2^2P(Y=2)=(0)(15/28)+(1)(3/7)+(4)(1/28)=4/7=0.5714$.
2b. We have $\mathbb{E}(Y^2)=\mathbb{E}((Y_1+Y_2)^2)=\mathbb{E}(Y_1^2)+2\mathbb{E}(Y_1Y_2)+\mathbb{E}(Y_2^2)$. We note $\mathbb{E}(Y_1^2)=\mathbb{E}(Y_1)=1/4$, and $\mathbb{E}(Y_2^2)=\mathbb{E}(Y_2)=1/4$, and $\mathbb{E}(Y_1Y_2)=(2/8)(1/7)=1/28$. Thus $\mathbb{E}(Y^2)=1/4+(2)(1/28)+1/4=4/7$.
2c. We have $\mathrm{Var}(Y)=\mathbb{E}(Y^2)-(\mathbb{E}(Y))^2=4/7-(1/2)^2=9/28=0.3214$.

3a. The mass of X is $P(X=0)=1/3$; $P(X=1)=1/2$; $P(X=2)=1/6$. Thus, we get $\mathbb{E}(X^2)=0^2P(X=0)+1^2P(X=1)+2^2P(X=2)=(0)(1/3)+(1)(1/2)+(4)(1/6)=7/6=1.1667$.
3b. We have $\mathbb{E}(X^2)=\mathbb{E}((X_1+X_2)^2)=\mathbb{E}(X_1^2)+2\mathbb{E}(X_1X_2)+\mathbb{E}(X_2^2)$. We note $\mathbb{E}(X_1^2)=\mathbb{E}(X_1)=1/3$ and $\mathbb{E}(X_2^2)=\mathbb{E}(X_2)=1/2$, and $\mathbb{E}(X_1X_2)=(1/3)(1/2)=1/6$. Thus $\mathbb{E}(X^2)=1/3+(2)(1/3)(1/2)+1/2=7/6=1.1667$.
3c. We have $\mathrm{Var}(X)=\mathbb{E}(X^2)-(\mathbb{E}(X))^2=7/6-(5/6)^2=17/36=0.4722$.
3d. Since the X_j's are independent, $\mathrm{Var}(X)=\mathrm{Var}(X_1+X_2)=\mathrm{Var}(X_1)+\mathrm{Var}(X_2)$. We have $\mathrm{Var}(X_1)=\mathbb{E}(X_1^2)-(\mathbb{E}(X_1))^2=1/3-(1/3)^2=2/9$, and $\mathrm{Var}(X_2)=\mathbb{E}(X_2^2)-(\mathbb{E}(X_2))^2=1/2-(1/2)^2=1/4$, so $\mathrm{Var}(X)=2/9+1/4=17/36=0.4722$.

4a. The mass of X is $P(X=j)=\binom{3}{j}\binom{6}{5-j}/\binom{9}{5}$ for $0\leq j\leq 3$, so we get $\mathbb{E}(X^2)=0^2P(X=0)+1^2P(X=1)+2^2P(X=2)+3^2P(X=3)=(0)(1/21)+(1)(5/14)+(4)(10/21)+(9)(5/42)=10/3=3.3333$.
4b. We have $\mathbb{E}(X^2)=\mathbb{E}((X_1+\cdots+X_5)^2)$, which has 20 terms of the form $\mathbb{E}(X_iX_j)$ (for $i\neq j$) and 5 terms of the form $\mathbb{E}(X_j^2)$. We have $\mathbb{E}(X_iX_j)=(3/9)(2/8)=1/12=0.0833$. Also, since indicators only take on values 0 or 1, then $\mathbb{E}(X_j^2)=\mathbb{E}(X_j)=3/9=1/3=0.3333$. Thus $\mathbb{E}(X^2)=(20)(1/12)+(5)(1/3)=(20)(0.0833)+(5)(0.3333)=10/3=3.3333$.
4c. We have $\mathbb{E}(X^2)=\mathbb{E}((Y_1+Y_2+Y_3)^2)=3\mathbb{E}(Y_1^2)+6\mathbb{E}(Y_1Y_2)$. We note $\mathbb{E}(Y_1^2)=\mathbb{E}(Y_1)=5/9=0.5556$, and $\mathbb{E}(Y_1Y_2)=(5/9)(4/8)=5/18=0.2778$. Thus $\mathbb{E}(X^2)=(3)(5/9)+(6)(5/18)=(3)(0.5556)+(6)(0.2778)=10/3=3.3333$.
4d. We have $\mathrm{Var}(X)=\mathbb{E}(X^2)-(\mathbb{E}(X))^2=10/3-(5/3)^2=5/9=0.5556$.