STAT/MA 41600
In-Class Problem Set #16: September 25, 2015
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Problem Set 16 Answers

1a. We observe that $P(X=Y)=\sum_{j=1}^{\infty}P(X=Y=j)=\sum_{j=1}^{\infty}(3/4)^{j-1}(1/4)(1/3)^{j-1}(2/3)$, where the last equality holds since X and Y are independent, so their joint probability mass function is a product of their probability mass functions. We simplify to obtain $P(X=Y)=(1/4)(2/3)\sum_{j=1}^{\infty}((3/4)(1/3))^{j-1}=\frac{(1/4)(2/3)}{1-(3/4)(1/3)}=2/(12-3)=2/9$.

1b. We have

$$P(X>Y)=\sum_{y=1}^{\infty}\sum_{x=y+1}^{\infty}P(X=x,Y=y)=\sum_{y=1}^{\infty}\sum_{x=y+1}^{\infty}(3/4)^{x-1}(1/4)(1/3)^{y-1}(2/3).$$

We can factor the $(1/4)(1/3)^{y-1}(2/3)$ out of the inner sum, to obtain

$$P(X>Y)=\sum_{y=1}^{\infty}(1/4)(1/3)^{y-1}(2/3)\sum_{x=y+1}^{\infty}(3/4)^{x-1}=\sum_{y=1}^{\infty}(1/4)(1/3)^{y-1}(2/3)\frac{(3/4)^y}{1-3/4}.$$

This simplifies to $P(X>Y)=(2/3)(3/4)\sum_{y=1}^{\infty}(1/4)^{y-1}=(2/3)(3/4)/(1-1/4)=2/3$.

Alternative (easier) method. Let us treat the outcomes that yield the values of X and Y as two sequences of independent trials that are performed at the same time, and see which sequence of trials succeeds first. We can effectively ignore all of the trials at the beginning in which both trials fail, and we focus on the first trial in which one or the other trial succeeds. The probability that one or the other (or both) trials succeed is $(3/4)(2/3)+(1/4)(1/3)+(1/4)(2/3)$. Thus, $P(X=Y)=\frac{(1/4)(2/3)}{(3/4)(2/3)+(1/4)(1/3)+(1/4)(2/3)}=2/9$. Similarly, $P(X>Y)=\frac{(3/4)(2/3)}{(3/4)(2/3)+(1/4)(1/3)+(1/4)(2/3)}=2/3$.

2a. We note $P(X+Y=5)=P(X=4,Y=1)+P(X=3,Y=2)+P(X=2,Y=3)+P(X=1,Y=4)$ but each of these four terms is equal to $(1/5)^3(4/5)(4/5)=16/3125$, because each term corresponds to a sequence of trials that is performed until the second success, i.e., which has 3 failures and 1 success within the first 4th trials, and then a success on the 5th trial. So $P(X+Y=5)=(4)(16/3125)=64/3125$.

2b. Each of X and Y has variance $(1/5)/(4/5)^2=5/16$. Since X and Y are independent, this yields $\mathrm{Var}(2Y-3X)=2^2\,\mathrm{Var}(Y)+(-3)^2\,\mathrm{Var}(X)=4(5/16)+9(5/16)=65/16=4.0625$.

3a. We have $P(A)=P(X=2)+P(X=4)+P(X=6)+P(X=8)=(2/3)(1/3)+(2/3)^3(1/3)+(2/3)^5(1/3)+(2/3)^7(1/3)+\cdots=(2/3)(1/3)(1+(2/3)^2+(2/3)^4+(2/3)^6+\cdots)=(2/3)(1/3)(1+4/9+(4/9)^2+(4/9)^3+\cdots)=(2/3)(1/3)\frac{1}{1-4/9}=2/5$.

3b. No, A and B are not independent, because B is a subset of A. (If one event is a subset of the other, they are dependent, unless one of them is $\emptyset$ or S.)

4a. We have $P(X>10)=(2/5)^{10}$.

4b. We have $P(X+Y>10)=(2/5)^{10}+\binom{10}{1}(2/5)^9(3/5)$.

4c. We have $P(X+Y+Z>10)=(2/5)^{10}+\binom{10}{1}(2/5)^9(3/5)+\binom{10}{2}(2/5)^8(3/5)^2$.