**Problem Set 18 Answers**

**1a.** Since $X$ and $Y$ are independent Poisson random variables, then $Z$ is a Poisson random variable too. We have $E(Z) = E(X + Y) = E(X) + E(Y) = 2 + 2 = 4$. We have $P(Z \leq 3) = p_Z(0) + p_Z(1) + p_Z(2) + p_Z(3) = e^{-4} + 4e^{-4} + 8e^{-4} + \frac{32}{3}e^{-4} = 0.0183 + 0.0733 + 0.1465 + 0.1954 = 0.4335$.

**1b.** We calculate a few values of the probability mass function of $X$, and we find that $p_X(x)$ attains its maximum when $x = 5$; indeed, we have $p_X(5) = e^{-5.3}(5.3)^5/5! = 0.1740$.

**2a.** We have

$$P(X > 4 \mid X > 2) = \frac{P(X > 4 \cap X > 2)}{P(X > 2)} = \frac{P(X > 4)}{1 - P(X \leq 2)} = \frac{1 - 0.1108 - 0.2438 - 0.2681 - 0.1966 - 0.1082}{1 - 0.1108 - 0.2438 - 0.2681} = 0.1922$$

**2b.** We have

$$P(X \leq 1 \mid X \leq 3) = \frac{P(X \leq 1 \cap X \leq 3)}{P(X \leq 3)} = \frac{P(X \leq 1)}{1 - 0.1108 - 0.2438 - 0.2681 - 0.1966} = 0.433$$

**3a.** The exact expression is \( \binom{5000000}{4} \left( \frac{1}{2000000} \right)^4 \left( \frac{1999999}{2000000} \right)^{5000000 - 4} = 0.133601909 \ldots \) (You do not need this last number in your answer; it takes a computer to approximate the answer.)

**3b.** The actual number of winners is a Binomial random variable with $n = 5000000$ and $p = 1/2000000$. So $n$ is large and $np(1 - p)$ is roughly $5/2$ which is a moderate size number, i.e., not too far from 1. So the number of winners is approximately Poisson with $\lambda = np = 5/2$. So the probability of 4 winners is approximately $e^{-5/2}(5/2)^4/4! = 0.133601886 \ldots$.

**4.** We have

$$E((X)(X - 1)(X - 2)) = \sum_{x=0}^{\infty} (x)(x - 1)(x - 2) e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= \sum_{x=3}^{\infty} (x)(x - 1)(x - 2) e^{-\lambda} \frac{\lambda^x}{x!} \quad \text{because } x = 0, 1, 2 \text{ terms are themselves 0}$$

$$= \sum_{x=3}^{\infty} e^{-\lambda} \frac{\lambda^x}{(x - 3)!} \quad \text{divide out by } x \text{ and } x - 1 \text{ and } x - 2$$

$$= \lambda^3 e^{-\lambda} \sum_{x=3}^{\infty} \frac{\lambda^{x-3}}{(x - 3)!} \quad \text{factor out } e^{-\lambda} \text{ and } \lambda^3$$

$$= \lambda^3 e^{-\lambda} \left( \frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \ldots \right)$$

$$= \lambda^3 e^{-\lambda}$$

$$= \lambda^3$$