STAT/MA 41600
In-Class Problem Set #24: October 14, 2015
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1a. We have $P(0.5 < X < 2.5) = \int_{0.5}^{2.5} \frac{2}{3} e^{-(2/3)x}\,dx = e^{-1/3} - e^{-5/3} = 0.5277$.
1b. We have $P(X = 2.5) = \int_{2.5}^{2.5} \frac{2}{3} e^{-(2/3)x}\,dx = 0$. In general, the probability that a continuous random variable takes on a specific value is always 0, i.e., $P(X = x) = 0$ for any continuous random variable X and any specific value x. The reasoning is always the same: It is equivalent to integrating a probability density function over an interval with no length at all. In this case, we integrate from 2.5 to 2.5, so we just get exactly 0.
1c. We have $F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx$. So $F_X(a) = 0$ for $a \leq 0$. For $a > 0$, we get $F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx = \int_0^a \frac{2}{3} e^{-(2/3)x}\,dx = 1 - e^{-(2/3)a}$. So altogether we have

$$F_X(x) = \begin{cases} 1 - e^{-(2/3)x} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

2a. The positive constant is 1/12 because we need to have $\int_8^{20} 1/12\,dx = 1$.
2b. We have $P(10 \leq X \leq 15) = \int_{10}^{15} 1/12\,dx = (15-10)/12 = 5/12$. Since we are integrating a constant, we can also just take the length of $[10, 15]$ and multiply by the constant, which is 1/12, i.e., which is 1 over the length of $[8, 20]$. So the desired probability is 5/12.
2c. We have $F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx$. So $F_X(a) = 0$ for $a < 8$, and $F_X(a) = 1$ for $a > 20$. For $8 \leq a \leq 20$, we get $F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx = \int_8^a 1/12\,dx = (a-8)/12$. So altogether we have

$$F_X(x) = \begin{cases} 0 & \text{for } x < 8, \\ (x-8)/12 & \text{for } 8 \leq x \leq 20, \\ 1 & \text{for } x > 20. \end{cases}$$

3a. The 25th percentile is the value of a such that $P(X \leq a) = 1/4$, i.e., such that $F_X(a) = 1/4$. So we have $1 - e^{-5a} = 1/4$, i.e., $e^{-5a} = 3/4$, so $-5a = \ln(3/4)$, and finally we get $a = -(1/5)\ln(3/4) = 0.0575$.
3b. The 50th percentile is found similarly: $1 - e^{-5a} = 1/2$, so $-5a = \ln(1/2)$, and finally we get $a = -(1/5)\ln(1/2) = 0.1386$.
3c. The 75th percentile is found similarly: $1 - e^{-5a} = 3/4$, so $-5a = \ln(1/4)$, and finally we get $a = -(1/5)\ln(1/4) = 0.2773$.
4ab.