STAT/MA 41600
In-Class Problem Set #26: October 19, 2015
Solutions by Mark Daniel Ward

1a. Here X and Y are dependent. Perhaps the easiest way to see this is that their domain is not rectangular shaped (it is like a triangle shape).
1b. We have $P(X \le 1) = \int_0^1 \int_0^{2x} \frac{1}{8}xy\,dy\,dx = \int_0^1 \frac{1}{4}x^3\,dx = 1/16$.
1c. The density of X is $f_X(x) = \int_0^{2x} \frac{1}{4}x^3\,dx = \frac{1}{4}x^3$ for $0 < x < 2$, and $f_X(x) = 0$ otherwise.
1d. Yes! We have $P(X \le 1) = \int_0^1 \frac{1}{4}x^3\,dx = 1/16$.

2a. We have $\int_0^\infty \int_{2x}^\infty 10e^{-3x-2y}\,dy\,dx = \int_0^\infty 5e^{-7x}\,dx = 5/7$.
2b. We have $f_X(x) = \int_x^\infty 10e^{-3x-2y}\,dy = 5e^{-5x}$ for $x > 0$, and $f_X(x) = 0$ otherwise.

3a. Yes, X and Y are independent. Their density is defined in a rectangular region, and it can be factored into x and y parts.
3b. We have $f_X(x) = \int_0^6 \frac{1}{225}(5-x)(6-y)\,dy = \frac{2}{25}(5-x)$, for $0 \le x \le 5$, and $f_X(x) = 0$ otherwise.
3c. We have $f_Y(y) = \int_0^5 \frac{1}{225}(5-x)(6-y)\,dx = \frac{1}{18}(6-y)$, for $0 \le y \le 6$, and $f_Y(y) = 0$ otherwise.

4a. For $z > 0$, we have $P(Z \ge z) = P(X \ge z \;\&\; Y \ge z) = P(X \ge z)P(Y \ge z) = (\int_z^\infty 3e^{-3x}\,dx)(\int_z^\infty 5e^{-5y}\,dy) = e^{-3z}e^{-5z} = e^{-8z}$. Thus $F_Z(z) = P(Z \le z) = 1 - e^{-8z}$ for $z > 0$. So $f_Z(z) = 8e^{-8z}$ for $z > 0$, and $f_Z(z) = 0$ otherwise.
4b. We have $P(Z > 1/10) = \int_{1/10}^\infty 8e^{-8z}\,dz = e^{-4/5} = 0.4493$.