STAT/MA 41600
In-Class Problem Set #27: October 21, 2015
Solutions by Mark Daniel Ward

1a. For $x > 0$, we have $f_X(x) = \int_x^\infty 70e^{-3x-7y}\,dy = 10e^{-10x}$; for $x \le 0$, we have $f_X(x) = 0$.
1b. For $x > 0$, the conditional density is $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{70e^{-3x-7y}}{10e^{-10x}} = 7e^{7x-7y}$.
1c. We have $f_{Y|X}(y \mid \frac{1}{10}) = 7e^{7(1/10)-7y}$, which is nonnegative, and $\int_{1/10}^\infty 7e^{7(1/10)-7y}\,dy = 1$.
1d. We have $P(Y > 1/4 \mid X = 1/10) = \int_{1/4}^\infty f_{Y|X}(y \mid \frac{1}{10}) = e^{-21/20} = 0.3499$.

2a. We don't need $f_{Y|X}(y \mid x)$ for this part at all. Instead, we use the basic definition of conditional probability from Problem Set 4 (second week of class). We need to compute $P(Y > 1/4 \mid X > 1/10) = \frac{P(Y>1/4\ \&\ X>1/10)}{P(X>1/10)}$. For the numerator, $P(Y > 1/4\ \&\ X > 1/10) = \int_{1/4}^\infty \int_{1/10}^y 70e^{-3x-7y}\,dx\,dy = \int_{1/4}^\infty \left(\frac{70}{3}e^{-(3/10)-7y} - \frac{70}{3}e^{-10y}\right)dy = \frac{10}{3}e^{-41/20} - \frac{7}{3}e^{-5/2} = 0.2376$. For the denominator, $P(X > 1/10) = \int_{1/10}^\infty \int_x^\infty 70e^{-3x-7y}\,dy\,dx = \int_{1/10}^\infty 10e^{-10x}\,dx = e^{-1} = 0.3679$. So we get $P(Y > 1/4 \mid X > 1/10) = \frac{P(Y>1/4\ \&\ X>1/10)}{P(X>1/10)} = 0.2376/0.3679 = 0.6458$.
2b. We need to compute $P(Y < 1/3 \mid X > 1/10) = \frac{P(Y<1/3\ \&\ X>1/10)}{P(X>1/10)}$. For the numerator, $P(Y < 1/3\ \&\ X > 1/10) = \int_{1/10}^{1/3}\int_{1/10}^y 70e^{-3x-7y}\,dx\,dy = \int_{1/10}^{1/3}\left(\frac{70}{3}e^{-(3/10)-7y} - \frac{70}{3}e^{-10y}\right)dy = e^{-1} - \frac{10}{3}e^{-79/30} + \frac{7}{3}e^{-10/3} = 0.2117$. The denominator is $P(X > 1/10) = e^{-1} = 0.3679$, just as in part **2a**. So we get $P(Y < 1/3 \mid X > 1/10) = \frac{P(Y<1/3\ \&\ X>1/10)}{P(X>1/10)} = 0.2117/0.3679 = 0.575$.

3a. For $0 \le x \le 2$, we have $f_X(x) = \int_0^{8-4x} 1/8\,dy = (8-4x)/8$. Therefore $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1/8}{(8-4x)/8} = \frac{1}{8-4x}$ for (x,y) in the triangle, i.e., $f_{Y|X}(y \mid x) = \frac{1}{8-4x}$ if $0 < y < 8-4x$, and $f_{Y|X}(y \mid x) = 0$ otherwise.
3b. We have $f_{Y|X}(y \mid 1/2) = \frac{1}{8-4(1/2)} = 1/6$. Thus $P(Y \le 4 \mid X = 1/2) = \int_0^4 1/6\,dy = 4/6 = 2/3$.
3c. We have $P(Y \le 4 \mid X \le 1/2) = \frac{P(Y\le 4\ \&\ X\le 1/2)}{P(X\le 1/2)}$. Both the numerator and denominator can be calculated by ratios of areas, since the joint density is constant. So we calculate $P(Y \le 4 \mid X \le 1/2) = \frac{P(Y\le 4\ \&\ X\le 1/2)}{P(X\le 1/2)} = \frac{2/8}{(7/2)/8} = \frac{2}{7/2} = 4/7$.

4a. For $0 \le x \le 5$, we have $f_X(x) = \int_0^{5-x} 2/25\,dy = (2/25)(5-x)$. So we get $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2/25}{(2/25)(5-x)} = \frac{1}{5-x}$.
4b. For $0 \le x \le c$, we have $f_X(x) = \int_0^{c-x} 2/c^2\,dy = (2/c^2)(c-x)$. So we get $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2/c^2}{(2/c^2)(c-x)} = \frac{1}{c-x}$.