1a. For $x > 0$, we have $f_X(x) = \int_x^\infty 70e^{-3x-y} \, dy = 10e^{-10x}$; for $x \leq 0$, we have $f_X(x) = 0$.
1b. For $x > 0$, the conditional density is $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{70e^{-3x-y}}{10e^{-10x}} = 7e^{7x-7y}$.
1c. We have $f_{Y|X}(y \mid \frac{1}{10}) = 7e^{7(1/10)-7y}$, which is nonnegative, and $\int_{1/10}^\infty 7e^{7(1/10)-7y} \, dy = 1$.
1d. We have $P(Y > 1/4 \mid X = 1/10) = \int_{1/10}^\infty f_{Y|X}(y \mid \frac{1}{10}) = e^{-21/20} = 0.3499$.

2a. We don’t need $f_{Y|X}(y \mid x)$ for this part at all. Instead, we use the basic definition of conditional probability from Problem Set 4 (second week of class). We need to compute $P(Y > 1/4 \mid X > 1/10) = \frac{P(Y > 1/4 \mid X > 1/10)}{P(X > 1/10)}$. For the numerator, $P(Y > 1/4 \land X > 1/10) = \int_{1/10}^\infty \int_{1/4}^y 70e^{-3x-y} \, dx \, dy = \int_{1/10}^\infty \left( \frac{70}{3} e^{-(3/10)-7y} - \frac{70}{3} e^{-10y} \right) dy = \frac{10}{3} e^{-41/20} - \frac{7}{3} e^{-5/2} = 0.2376$. For the denominator, $P(X > 1/10) = \int_{1/10}^\infty \int_{1/10}^\infty 70e^{-3x-y} \, dy \, dx = \int_{1/10}^\infty 10e^{-10x} \, dx = e^{-1} = 0.3679$. So we get $P(Y > 1/4 \mid X > 1/10) = \frac{P(Y > 1/4 \land X > 1/10)}{P(X > 1/10)} = 0.2376/0.3679 = 0.6458$.

2b. We need to compute $P(Y < 1/3 \mid X > 1/10) = \frac{P(Y < 1/3 \land X > 1/10)}{P(X > 1/10)}$. For the numerator, $P(Y < 1/3 \land X > 1/10) = \int_{1/10}^{1/3} \int_{1/10}^y 70e^{-3x-y} \, dx \, dy = \int_{1/10}^{1/3} \left( \frac{70}{3} e^{-(3/10)-7y} - \frac{70}{3} e^{-10y} \right) dy = e^{-1} - \frac{10}{3} e^{-79/30} + \frac{7}{3} e^{-10/3} = 0.2117$. The denominator is $P(X > 1/10) = e^{-1} = 0.3679$, just as in part 2a. So we get $P(Y < 1/3 \mid X > 1/10) = \frac{P(Y < 1/3 \land X > 1/10)}{P(X > 1/10)} = 0.2117/0.3679 = 0.575$.

3a. For $0 \leq x \leq 2$, we have $f_X(x) = \int_0^{8-4x} 1/8 \, dy = (8 - 4x)/8$.
3b. We have $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1/8}{8-4x}/8 = \frac{1}{8-4(1/2)} = 1/6$. Thus $P(Y \leq 4 \mid X = 1/2) = \int_{1/2}^4 \frac{1}{6} \, dy = 4/6 = 2/3$.
3c. We have $P(Y \leq 4 \mid X \leq 1/2) = \frac{P(Y \leq 4 \land X \leq 1/2)}{P(X \leq 1/2)}$. Both the numerator and denominator can be calculated by ratios of areas, so the joint density is constant. So we calculate $P(Y \leq 4 \mid X \leq 1/2) = \frac{P(Y \leq 4 \land X \leq 1/2)}{P(X \leq 1/2)} = \frac{2/8}{(7/2)/8} = \frac{2}{7/2} = 4/7$.

4a. For $0 \leq x \leq 5$, we have $f_X(x) = \int_0^{5-x} 2/25 \, dy = (2/25)(5-x)$. So we get $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2/25}{(2/25)(5-x)} = \frac{1}{5-x}$.
4b. For $0 \leq x \leq c$, we have $f_X(x) = \int_0^{c-x} 2/c^2 \, dy = (2/c^2)(c-x)$. So we get $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2/c^2}{(2/c^2)(c-x)} = \frac{1}{c-x}$.