

STAT/MA 41600
In-Class Problem Set #29: October 26, 2015
Solutions by Mark Daniel Ward

1a. One method is that we can compute

$$\mathbb{E}(X^2) = \int_0^\infty \int_x^\infty (x^2)(70e^{-3x-7y}) dy dx = \int_0^\infty (x^2)(70e^{-3x}) \int_x^\infty e^{-7y} dy dx = \int_0^\infty (x^2)(70e^{-3x})(1/7)e^{-7x} dx,$$

which simplifies to

$$\mathbb{E}(X^2) = \int_0^\infty (x^2)(10e^{-10x}) dx = (x^2)(-e^{-10x}) \Big|_{x=0}^\infty - \int_0^\infty (-e^{-10x})(2x) dx = 0 + 2 \int_0^\infty xe^{-10x} dx$$

We already computed (in **1** of the last problem set): $10 \int_0^\infty xe^{-10x} dx = 1/10$, and thus $\mathbb{E}(X^2) = 2 \int_0^\infty xe^{-10x} dx = (2/10)(1/10) = 2/100$.

1b. We have $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 2/100 - (1/10)^2 = 1/100$.

2a. We have $\mathbb{E}(XY) = \int_0^5 \int_0^8 (xy)(1/40) dy dx = \int_0^5 (x)(4/5) dx = 10$.

2b. Yes, X and Y are independent. Their joint density $1/40$ can be factored into $1/5$ and $1/8$, and the joint density is defined on a rectangle.

2c. We have $\mathbb{E}(X) = \int_0^5 (x)(1/5) dx = 5/2$.

2d. We have $\mathbb{E}(Y) = \int_0^8 (y)(1/8) dy = 4$.

Thus, we can use parts **2b**, **2c**, **2d** to double check that $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) = (5/2)(4) = 10$. (We emphasize that we can only multiply the expected values this way because the X and Y are independent.)

3a. One method is that we can compute

$$\mathbb{E}(X^2) = \int_0^2 \int_0^{8-4x} (x^2)(1/8) dy dx = \int_0^2 (x^2)(1/8) \int_0^{8-4x} 1 dy dx = \int_0^2 (x^2)(1/8)(8-4x) dx,$$

which simplifies to

$$\mathbb{E}(X^2) = \int_0^2 (x^2) \left(\frac{8-4x}{8} \right) dx = 2/3.$$

FYI, if you decided (instead) to just directly use the density of X , namely, $f_X(x) = \frac{8-4x}{8}$ for $0 \leq x \leq 2$, we get exactly the line above, $\mathbb{E}(X^2) = \int_0^2 (x^2) \left(\frac{8-4x}{8} \right) dx = 2/3$.

3b. One method is that we can compute

$$\mathbb{E}(XY) = \int_0^2 \int_0^{8-4x} (xy)(1/8) dy dx = \int_0^2 (x)(1/8) \int_0^{8-4x} y dy dx = \int_0^2 (x)(1/8)(8x^2 - 32x + 32) dx = 4/3.$$

You could also have changed the order of integration and the bounds, as another possible method of solution.

4a. We have

$$\mathbb{E}(Y^2) = \int_0^\infty (y^2)(5e^{-5y}) dy = (y^2)(-e^{-5y}) \Big|_{y=0}^\infty - \int_0^\infty (2y)(-e^{-5y}) dy$$

which simplifies to $\mathbb{E}(Y^2) = (2) \int_0^\infty (y)(-e^{-5y}) dy$. We saw in **4a** from the previous problem set that $5 \int_0^\infty (y)(-e^{-5y}) dy = 1/5$, so it follows that $\mathbb{E}(Y^2) = (2/5)(1/5) = 2/25$.

4b. We have $\text{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 2/25 - (1/5)^2 = 1/25$.