STAT/MA 41600
In-Class Problem Set #31: October 28, 2015
Solutions by Mark Daniel Ward

1a. Since the joint density is constant, one possible method is to use the areas of the regions under study. The area of the whole region is 8, and the area where $X > Y$ is 6. Thus $P(X > Y) = 6/8 = 3/4$. Another possible method is to integrate: $\int_0^2 \int_y^{y+3} 1/8\,dx\,dy = \int_0^2 3/8\,dy = 3/4$.
1b. Since the joint density is constant, one possible method is to use the areas of the regions under study. The area of the whole region is 8, and the area where $X + Y \le 3$ is 4. Thus $P(X + Y \le 3) = 4/8 = 1/2$. Another possible method is to integrate: $\int_0^2 \int_0^{3-y} 1/8\,dx\,dy = \int_0^2 (3-y)/8\,dy = 1/2$.

2. One method is to write $Z = \max(X, Y)$. For $0 < z < 5$, we have $F_Z(a) = P(Z \le a) = a^2/25$, so $f_Z(z) = 2z/25$, and thus $\mathbb{E}(\max(X, Y)) = \mathbb{E}(Z) = \int_0^5 (z)(2z/25)\,dz = 10/3$.

Another method is to note that $\max(X, Y) = Y$ when $Y > X$, and $\max(X, Y) = X$ when $Y < X$. Thus, we get $\mathbb{E}(\max(X,Y)) = \int_0^5 \int_0^x (x)(1/25)\,dy\,dx + \int_0^5 \int_0^y (y)(1/25)\,dx\,dy = \int_0^5 x^2/25\,dx + \int_0^5 y^2/25\,dy = 5/3 + 5/3 = 10/3$.

3. Since the joint density is constant, perhaps the easiest method is to use the areas of the regions under study. The area of the whole region is 24, and the area where the butterfly is less than 1 unit from the point $(3, 2)$ is π. So the desired probability is $\pi/24 = 0.1309$.

4. We have

$$P(Y > X) = \int_0^2 \int_x^\infty (1/2)(e^{-y})\,dy\,dx = \int_0^2 (1/2)(e^{-x})\,dx = (1 - e^{-2})/2 = 0.4323.$$