STAT/MA 41600
In-Class Problem Set #32: October 30, 2015
Solutions by Mark Daniel Ward

1. We have $P(X > 1 \;\&\; Y > 1) = \int_1^\infty \int_1^\infty (2e^{-2x})(3e^{-3y})\,dy\,dx$. Or, since X and Y are independent, you might choose to write $P(X > 1 \;\&\; Y > 1) = P(X > 1)P(Y > 1) = (\int_1^\infty 2e^{-2x}\,dx)(\int_1^\infty 3e^{-3y}\,dy)$. Either way, you get $(e^{-2})(e^{-3}) = e^{-5} = 0.006738$.

2a. We have $P(X > 4 \mid X > 3) = \frac{P(X>4 \;\&\; X>3)}{P(X>3)} = \frac{P(X>4)}{P(X>3)} = \frac{e^{-2(4)}}{e^{-2(3)}} = \frac{e^{-8}}{e^{-6}} = e^{-2}$.
2b. Yes, we also have $P(X > 1) = e^{-2(1)} = e^{-2}$.
2c. We have $P(U > 4 \mid U > 3) = \frac{P(U>4 \;\&\; U>3)}{P(U>3)} = \frac{P(U>4)}{P(U>3)} = \frac{6/10}{7/10} = 6/7$. This is not equal to $P(U > 1)$, because $P(U > 1) = 9/10$.

3a. Since the X_j's are independent, then

$$P(X_j \geq 20 \text{ for all } j) = P(X_1 > 20)P(X_2 > 20)P(X_3 > 20)P(X_4 > 20)P(X_5 > 20),$$

and all 5 of the terms on the right hand side are equal. For each j, we have $P(X_j > 20) = \int_{20}^\infty \frac{1}{8}e^{-(1/8)x}\,dx = e^{-(1/8)(20)} = e^{-2.5}$. Thus $P(X_j \geq 20 \text{ for all } j) = (e^{-2.5})^5 = e^{-12.5} = 0.000003727$.

Since the X_j's are independent, then

$$\begin{aligned} P(X_j \geq 20 \text{ for at least one } j) &= 1 - P(X_j \leq 20 \text{ for all } j) \\ &= 1 - P(X_1 \leq 20)P(X_2 \leq 20)P(X_3 \leq 20)P(X_4 \leq 20)P(X_5 \leq 20). \end{aligned}$$

For each j, we have $P(X_j \leq 20) = \int_0^{20} \frac{1}{8}e^{-(1/8)x}\,dx = 1 - e^{-(1/8)(20)} = 1 - e^{-2.5}$. Thus $P(X_j \geq 20 \text{ for at least one } j) = 1 - (1 - e^{-2.5})^5 = 0.3484$.
3b. Since the Y_j's are independent, then

$$P(Y_j \geq 20 \text{ for all } j) = P(Y_1 > 20)P(Y_2 > 20) \cdots P(Y_n > 20),$$

and all n of the terms on the right hand side are equal. For each j, we have exactly as before $P(Y_j > 20) = e^{-2.5}$. Thus $P(Y_j \geq 20 \text{ for all } j) = (e^{-2.5})^n = e^{-2.5n}$.

Since the Y_j's are independent, then

$$\begin{aligned} P(Y_j \geq 20 \text{ for at least one } j) &= 1 - P(Y_j \leq 20 \text{ for all } j) \\ &= 1 - P(Y_1 \leq 20)P(Y_2 \leq 20) \cdots P(Y_n \leq 20). \end{aligned}$$

For each j, we have exactly as before $P(Y_j \leq 20) = 1 - e^{-2.5}$. Thus $P(Y_j \geq 20 \text{ for at least one } j) = 1 - (1 - e^{-2.5})^n$.

4. We compute $P(2X < Y) = \int_0^\infty \int_{2x}^\infty \frac{1}{5}e^{-(1/5)x}\frac{1}{5}e^{-(1/5)y}\,dy\,dx = \int_0^\infty \frac{1}{5}e^{-(3/5)x}\,dx = 1/3$.