We have \( P(X + Y < 12) = \int_0^{12} \int_0^{12-x} (1/5)e^{-(1/5)x}(1/5)e^{-(1/5)y} dy dx \). For the inner integral, we focus on the \( x \) part of the integrand, and we get: \( \int_0^{12-x} (1/5)e^{-(1/5)y} dy = 1 - e^{-(12/5)+(1/5)x} \). So then we have \( P(X + Y < 12) = \int_0^{12} (1/5)e^{-(1/5)x}(1 - e^{-(12/5)+(1/5)x}) dx = \int_0^{12} (1/5)e^{-(1/5)x} dx - \int_0^{12} (1/5)e^{-(1/5)y} dy = 1 - e^{-(12/5)} - (12/5)e^{-(12/5)} = 1 - (17/5)e^{-(12/5)} = 0.6916. \)

1b. The calculation is the same, but using “a” instead of 12.

We have \( P(X + Y < a) = \int_0^a \int_0^{a-x} (1/5)e^{-(1/5)x}(1/5)e^{-(1/5)y} dy dx \). For the inner integral, we focus on the \( x \) part of the integrand, and we get: \( \int_0^{a-x} (1/5)e^{-(1/5)y} dy = 1 - e^{-(1/5)a+(1/5)x} \). So then we have \( P(X + Y < a) = \int_0^a (1/5)e^{-(1/5)x}(1 - e^{-(1/5)a+(1/5)x}) dx = \int_0^a (1/5)e^{-(1/5)x} dx - \int_0^a (1/5)e^{-(1/5)a} dx = 1 - e^{-(1/5)a} - (a/5)e^{-(1/5)a} = 1 - (1 + \frac{a}{5})e^{-(1/5)a}. \)

1c. Yes! If we use \( a = 12 \) in our answer to 1b, we get \( P(X + Y < 12) = 1 - (1 + \frac{12}{5})e^{-(1/5)12} = 1 - (17/5)e^{-(12/5)} = 0.6916. \)

2a. We have \( 0 < U < 1 \), and therefore \(-\infty < \ln U < 0\), so we conclude \( 0 < -\frac{1}{5}\ln U < \infty\). So \( X \) takes on positive real values.

2b. For \( a \leq 0 \), we have \( F_X(a) = 0 \). For \( a > 0 \), we compute \( F_X(a) = P(X \leq a) = P(-\frac{1}{5}\ln U \leq a) = P(\ln U \geq -5a) = P(U \geq e^{-5a}) = 1 - e^{-5a}. \)

2c. Using the CDF of \( X \) from 2b, we see that \( X \) is an Exponential random variable with \( \lambda = 5 \), i.e., with average 1/5.

3. We have \( P(X - Y > 1) = \int_0^{\infty} \int_{y+1}^{\infty} e^{-x}e^{-y} dx dy \). For the inner integral, we focus on the \( x \) part of the integrand, and we get: \( \int_{y+1}^{\infty} e^{-x} dx = e^{-(y+1)}. \) So then we have \( P(X - Y > 1) = \int_0^{\infty} e^{-y}e^{-(y+1)} dy = \int_0^{\infty} e^{-2y-1} dy = (1/2)e^{-1}. \)

Similarly (just switching the roles of \( x \) and \( y \)), we get \( P(Y - X > 1) = (1/2)e^{-1}. \)

So altogether we have \( P(|X - Y| > 1) = (1/2)e^{-1} + (1/2)e^{-1} = e^{-1} = 0.3679. \)

4a. We have \( P(X < 1) = F_X(1) = 1 - e^{-(1/2)(1)} = 0.3935. \)

4b. The probability is \( P(X < 1, Y < 1, Z < 1) = P(X < 1)P(Y < 1)P(Z < 1) = (1 - e^{-(1/2)(1)})^3 = 0.0609. \)

4c. The probability is \( P(X > 1, Y > 1, Z > 1) = P(X > 1)P(Y > 1)P(Z > 1) = (e^{-(1/2)(1)})^3 = 0.2231. \)

4d. The probability is \( 3(1 - e^{-(1/2)(1)})^2(e^{-(1/2)(1)})^2 = 0.4342. \)

4e. The probability is \( 3(1 - e^{-(1/2)(1)})^2(e^{-(1/2)(1)})^1 = 0.2817. \)

4f. The random variable \( V \) is a Binomial random variable with parameters \( n = 3 \) and \( p = 1 - e^{-1/2} = 0.3935. \)