STAT/MA 41600
In-Class Problem Set #33: November 4, 2015
Solutions by Mark Daniel Ward

1a. We have $P(X_1 + X_2 < a) = \int_0^a \int_0^{a-x_1} \lambda e^{-\lambda x_1} \lambda e^{-\lambda x_2}\, dx_2\, dx_1$. For the inner integral, we focus on the x_2 part of the integrand, and we get: $\int_0^{a-x_1} \lambda e^{-\lambda x_2}\, dx_2 = 1 - e^{-\lambda a + \lambda x_1}$. So then we have $P(X_1 + X_2 < a) = \int_0^a \lambda e^{-\lambda x_1}(1 - e^{-\lambda a + \lambda x_1})\, dx_1 = \int_0^a \lambda e^{-\lambda x_1}\, dx_1 - \int_0^a \lambda e^{-\lambda a}\, dx_1 = 1 - e^{-\lambda a} - \lambda a e^{-\lambda a} = 1 - (1 + \lambda a)e^{-\lambda a}$.

1b. We have $P(X_1 + X_2 + X_3 < a) = \int_0^a \int_0^{a-x_1} \int_0^{a-x_1-x_2} \lambda e^{-\lambda x_1} \lambda e^{-\lambda x_2} \lambda e^{-\lambda x_3}\, dx_3\, dx_2\, dx_1$. [You do not have to calculate this triple integral, but just FYI, $P(X_1 + X_2 + X_3 < a) = 1 - e^{-\lambda a}(1 + \lambda a + \frac{(\lambda a)^2}{2})$.]

2a. Yes! The random variables U and V are independent, because $f_{U,V}(u, v)$ can be factored into u and v parts.
2b. By symmetry, we have $f_U(u) = 3e^{-3u}$ for u positive, and $f_U(u) = 0$ otherwise.
2c. The random variable X is a Gamma random variable with parameters $r = 2$ and $\lambda = 3$, so the density of X is $f_X(x) = 9xe^{-3x}$ for $x > 0$, and $f_X(x) = 0$ otherwise.
2d. We have $P(X \leq 1/2) = F_X(1/2) = 1 - e^{-(3)(1/2)}(1 + (3)(1/2)) = 1 - (5/2)e^{-3/2}$. Or we could calculate $P(X \leq 1/2) = \int_0^{1/2} 9xe^{-3x}\, dx = 1 - (5/2)e^{-3/2}$.

3a. We have $P(U > V) + P(U = V) + P(U < V) = 1$, and $P(U = V) = 0$, and $P(U > V) = P(U < V)$, so it must be the case that $P(U > V) = 1/2$.
3b. The random variable Y is a Binomial random variable with parameters $n = 2$ and $p = 1 - e^{-(3)(1/10)} = 0.2592$.

4a. The random variable W is a Gamma random variable with $r = 3$ and $\lambda = 1/5$. The density of W is $f_W(w) = \frac{(1/5)^3 w^2}{2} e^{-(1/5)w}$ for $w > 0$, and $f_W(w) = 0$ otherwise.
4b. The variance of W is $\operatorname{Var}(W) = r/\lambda^2 = \frac{3}{(1/5)^2} = 75$.
4c. The variance of $W/60$ is $\operatorname{Var}(W/60) = 75(1/60)^2 = 1/48$.