STAT/MA 41600
In-Class Problem Set #39 part 2: November 23, 2015
Solutions by Mark Daniel Ward

1. Let $X_1 = 1$ if red appears on the red/green/blue die, or $X_1 = 0$ otherwise. Let $X_2 = 1$ if red appears on the red/blue die, or $X_2 = 0$ otherwise. So $X = X_1 + X_2$. It follows that $\mathrm{Var}(X) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2\,\mathrm{Cov}(X_1, X_2)$ We have $\mathrm{Var}(X_1) = 2/6 - (2/6)^2 = 2/9$, and $\mathrm{Var}(X_2) = 3/6 - (3/6)^2 = 1/4$, and $\mathrm{Cov}(X_1, X_2) = 0$ since X_1 and X_2 are independent. So altogether $\mathrm{Var}(X) = 2/9 + 1/4 + (2)(0) = 17/36$.

2. We can write $X = X_1 + \cdots + X_{10}$ where $X_j = 1$ if the jth pair has 1 red and 1 green, or $X_j = 0$ otherwise. Then $\mathbb{E}(X_j) = 10/19$ for each j. Also, $\mathrm{Var}(X) = \mathrm{Var}(X_1 + \cdots + X_{10}) = \sum_{i=1}^{10} \mathrm{Var}(X_i) + \sum_{i\neq j} \mathrm{Cov}(X_i, X_j)$. We have $\mathrm{Var}(X_i) = \mathbb{E}(X_i^2) - (\mathbb{E}(X_i))^2 = 10/19 - (10/19)^2 = 90/361$ for each i. Also $\mathrm{Cov}(X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j) = (10/19)(9/17) - (10/19)^2 = 10/6137$ for each $i \neq j$. So altogether we have $\mathrm{Var}(X) = (10)(90/361) + (90)(10/6137) = 16200/6137 = 2.64$.

3. Since the joint probability density function is constant, it must be $f_{X,Y}(x, y) = 2/25$ for x, y in the triangle, and $f_{X,Y}(x, y) = 0$ otherwise. We have $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. Also $\mathbb{E}(XY) = \int_0^5 \int_0^{5-x} (xy)(2/25)\,dy\,dx = 25/12$, and $\mathbb{E}(X) = \int_0^5 \int_0^{5-x} (x)(2/25)\,dy\,dx = 5/3$, and similarly $\mathbb{E}(Y) = 5/3$. So $\mathrm{Cov}(X, Y) = 25/12 - (5/3)^2 = -25/36$.

4. We have $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. We compute $\mathbb{E}(XY) = \int_0^1 \int_x^1 xye^{1-x}\,dy\,dx = \int_0^1 xe^{1-x} \int_x^1 y\,dy\,dx = \int_0^1 xe^{1-x}(1 - x^2)/2\,dx = \int_0^1 xe^{1-x}(1 - x^2)/2\,dx = \frac{e}{2} \int_0^1 e^{-x}(x - x^3)\,dx = 7 - (5/2)(e) = 0.2043$. Also we compute $\mathbb{E}(X) = \int_0^1 \int_x^1 xe^{1-x}\,dy\,dx = \int_0^1 xe^{1-x} \int_x^1 1\,dy\,dx = \int_0^1 xe^{1-x}(1 - x)\,dx = e \int_0^1 e^{-x}(x - x^2)\,dx = 3 - e = 0.2817$ and $\mathbb{E}(Y) = \int_0^1 \int_x^1 ye^{1-x}\,dy\,dx = \int_0^1 e^{1-x} \int_x^1 y\,dy\,dx = \int_0^1 e^{1-x}(1 - x^2)/2\,dx = \frac{e}{2} \int_0^1 e^{-x}(1 - x^2)\,dx = 2 - e/2 = 0.6409$. So we conclude that $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0.2043 - (0.2817)(0.6409) = 0.0238$.