STAT/MA 41600
In-Class Problem Set #42: December 4, 2015
Solutions by Mark Daniel Ward

1a. We have $f_{X_{(1)}}(x_1) = \binom{3}{0!1!2!}(\frac{1}{10})(\frac{x_1}{10})^0(1-\frac{x_1}{10})^2 = (\frac{3}{10})(1-\frac{x_1}{10})^2$ for $0 < x_1 < 10$, and $f_{X_{(1)}}(x_1) = 0$ otherwise.

1b. We have $f_{X_{(2)}}(x_2) = \binom{3}{1!1!1!}(\frac{1}{10})(\frac{x_2}{10})^1(1-\frac{x_2}{10})^1 = (\frac{3}{50})(x_2)(1-\frac{x_2}{10})$ for $0 < x_2 < 10$, and $f_{X_{(2)}}(x_2) = 0$ otherwise.

1c. We have $f_{X_{(3)}}(x_3) = \binom{3}{2!1!0!}(\frac{1}{10})(\frac{x_3}{10})^2(1-\frac{x_3}{10})^0 = \frac{3x_3^2}{1000}$ for $0 < x_3 < 10$, and $f_{X_{(3)}}(x_3) = 0$ otherwise.

2a. We have $\mathbb{E}(X_{(1)}) = \int_0^{10}(x_1)(\frac{3}{10})(1-\frac{x_1}{10})^2\,dx_1 = 5/2$.

2b. We have $\mathbb{E}(X_{(2)}) = \int_0^{10}(x_2)(\frac{3}{50})(x_2)(1-\frac{x_2}{10})\,dx_2 = 5$.

2c. We have $\mathbb{E}(X_{(3)}) = \int_0^{10}(x_3)(\frac{3x_3^2}{1000})\,dx_3 = 15/2$.

2d. Indeed, we get $5/2 + 5 + 15/2 = 15$, as we knew we must.

3a. We see that X_1 and X_2 each have density $(1/8)(4-x)$ for $0 < x < 4$, and therefore each have CDF $\int_0^a (1/8)(4-x)\,dx = (a/16)(8-a)$ for $0 < a < 4$. Therefore, we have

$$\begin{aligned} f_{X_1}(x_1) &= \binom{2}{0,1,1}(1/8)(4-x_1)((x_1/16)(8-x_1))^0(1-(x_1/16)(8-x_1))^1 \\ &= \left(\frac{1}{64}\right)(4-x_1)^3 \\ &= 1-(3/4)x_1+(3/16)x_1^2-(1/64)x_1^3 \end{aligned}$$

for $0 < x_1 < 4$.

3b. We have

$$\begin{aligned} f_{X_2}(x_2) &= \binom{2}{1,1,0}(1/8)(4-x_2)((x_2/16)(8-x_2))^1(1-(x_2/16)(8-x_2))^0 \\ &= \left(\frac{x_2}{64}\right)(4-x_2)(8-x_2) \\ &= (1/64)x_2^3-(3/16)x_2^2+(1/2)x_2 \end{aligned}$$

for $0 < x_2 < 4$.

4a. We have $\mathbb{E}(X_{(1)}) = \int_0^4 (x_1)(1-(3/4)x_1+(3/16)x_1^2-(1/64)x_1^3)\,dx_1 = 4/5$.

4b. We have $\mathbb{E}(X_{(2)}) = \int_0^4 (x_2)((1/64)x_2^3-(3/16)x_2^2+(1/2)x_2)\,dx_2 = 28/15$.

4c. Indeed, we get $4/5 + 28/15 = 8/3$, as we knew we must.