STAT/MA 41600
In-Class Problem Set #43: December 7, 2015
Solutions by Mark Daniel Ward

1a. We have $M_X(t) = \mathbb{E}(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-3}3^x}{x!} = e^{-3} \sum_{x=0}^{\infty} \frac{(e^t 3)^x}{x!} = e^{-3} e^{(3e^t)} = e^{3(e^t-1)}$

1b. We have $M'_X(t) = \frac{d}{dt} e^{3(e^t-1)} = (e^{3(e^t-1)})(3e^t)$, so $M'_X(0) = (e^{3(e^0-1)})(3e^0) = 3$.

2a. We compute $M_X(t) = \mathbb{E}(e^{tX}) = \int_0^{\infty} (e^{tx})(\frac{1}{15} e^{-x/15})\,dx = \frac{1/15}{(1/15)-t}$.

2b. We have $M'_X(t) = \frac{d}{dt} \frac{1/15}{(1/15)-t} = \frac{1/15}{((1/15)-t)^2}$. So $\mathbb{E}(X) = M'_X(0) = \frac{1/15}{((1/15)-0)^2} = 15$.

3a. From **2b**, we have $M'_X(t) = \frac{d}{dt} \frac{1/15}{(1/15)-t} = \frac{1/15}{((1/15)-t)^2}$. Taking another derivative with respect to t, we get $M''_X(t) = \frac{d}{dt} \frac{1/15}{((1/15)-t)^2} = (2)\frac{1/15}{((1/15)-t)^3}$. So $\mathbb{E}(X^2) = M''_X(0) = (2)\frac{1/15}{((1/15)-0)^3} = 2(15^2)$.

3b. We have $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 2(15^2) - (15)^2 = 15^2$. This is what we knew we would get for the answer, since for an Exponential random variable, the variance is equal to the square of the mean.

4a. All of the values $p_X(x) = P(X = x)$ are nonnegative, and we have $\sum_{x=0}^{3} (27/40)(1/3)^x = 0.675 + 0.225 + 0.075 + 0.025 = 1$. So $p_X(x)$ is a valid probability mass function.

4b. We compute $(0)(0.675) + (1)(0.225) + (2)(0.075) + (3)(0.025) = 0.45$.

4c. We have $M_X(t) = \mathbb{E}(e^{tX}) = \sum_{x=0}^{3} e^{tx} (\frac{27}{40})(1/3)^x = (\frac{27}{40}) \sum_{x=0}^{3} (e^t/3)^x = (\frac{27}{40}) \frac{1-(e^t/3)^4}{1-e^t/3}$.

4d. We have $M'_X(t) = \frac{d}{dt}(\frac{27}{40})\frac{1-(e^t/3)^4}{1-e^t/3} = (\frac{27}{40})\frac{(1-e^t/3)(-4(e^t/3)^3(1/3))-(1-(e^t/3)^4)(-e^t/3)}{(1-e^t/3)^2}$. So $\mathbb{E}(X) = M'_X(0) = (\frac{27}{40})\frac{(1-e^0/3)(-4(e^0/3)^3(1/3))-(1-(e^0/3)^4)(-e^0/3)}{(1-e^0/3)^2} = (\frac{27}{40})\frac{(2/3)(-4(1/3)^4)-(1-(1/3)^4)(-1/3)}{(2/3)^2} = 0.45$.