STAT/MA 41600
Midterm Exam 1 Answers
Friday, October 9, 2015
Solutions by Mark Daniel Ward

1a. We compute \( P(X = Y) = \sum_{n=1}^{\infty} P(X = Y = n) = \sum_{n=1}^{\infty} P(X = n)P(Y = n) = \sum_{n=1}^{\infty}(1/3)(2/3)^{n-1}(2/5)(3/5)^{n-1} = \frac{(1/3)(2/5)}{1-(2/3)(3/5)} = 2/9. \)

1b. We compute \( P(X > Y) = \sum_{n=1}^{\infty} P(X > Y = n) = \sum_{n=1}^{\infty} P(X > n)P(Y = n) = \sum_{n=1}^{\infty}(2/3)^{n}(2/5)(3/5)^{n-1} = \frac{(2/3)(2/5)}{1-(2/3)(3/5)} = 4/9. \)

2. Since \( X \) is a Negative Binomial random variable with \( r = 6 \) and \( p = 0.45 \) then \( E(X) = r/p = 40/3 = 13.3333 \) and \( \text{Var}(X) = rq/p^2 = 440/27 = 16.3363. \) Also \( P(X \geq 9) = 1 - P(X < 9) = 1 - P(X = 8) - P(X = 7) - P(X = 6) = 1 - \binom{9}{6} q^6 p^6 - \binom{9}{5} q^5 p^6 - \left(\frac{6}{11}\right) p^6 = 0.9115. \)

3a. Let \( X_j \) be a Bernoulli random variable that indicates, for the \( j \)th bear that is yellow or blue, whether that particular bear is selected before all of the red bears. So \( E[X_j] = P(X_j = 1) = 1/11. \) The total number of bears selected before the first red is \( X_1 + \cdots + X_{20}. \) Thus \( E(X_1 + \cdots + X_{20}) = E(X_1) + \cdots + E(X_{20}) = 1/11 + \cdots + 1/11 = 20/11 \approx 1.818. \)

3b. We compute \( E(X^2) = E(X_1 + \cdots + X_{20})^2 = 20E(X_1^2) + 380E(X_1X_2) = (20)(\frac{1}{11}) + (380)(\frac{1}{11})(\frac{1}{11}) = 250/33 \approx 7.576. \) So \( \text{Var}(X) = E(X^2) - (E(X))^2 = 250/33 - (20/11)^2 \approx 4.27. \)

4a. The exact expression is \( P(X = 4) = \binom{20000}{4}\binom{40000}{6}\binom{60000}{10} = 0.227625 \ldots \) (You did not have to put the decimal value, of course; it is probably way too large for your calculator.)

4b. Since \( X \) is approximately Binomial with \( n = 10 \) and \( p = M/N = 20000/60000 = 1/3, \) then \( P(X = 4) \) is approximately equal to \( \binom{10}{4}(1/3)^4(2/3)^6 = 0.227608 \ldots. \)

5a. Without loss of generality, place one red plate on the table. If all three red plates are to be in a cluster, this first red plate could be the far left, the middle, or the far right of the three in the eventual cluster (i.e., 3 possibilities). Each such possibility has probability \( (2/5)(1/4) = 2/20 \) of occurring. So the total probability is \( 2/20 + 2/20 + 2/20 = 6/20 = 3/10. \) [[Alternatively: Place one red plate. Then there are \( \binom{3}{1} = 10 \) equally likely ways remaining for the blue plates; in 3 of these ways, the blue plates are adjacent (and therefore the red plates are adjacent too), so the probability is \( 3/10. \)]]

5b. Place one red plate on the table. If all four red plates are to be in a cluster, this first red plate could be the far left, the middle left, the middle right, or the far right of the four in the eventual cluster (i.e., 4 possibilities). Each such possibility has probability \( (3/7)(2/6)(1/5) = 1/35 \) of occurring. So the total probability is \( 1/35 + 1/35 + 1/35 + 1/35 = 4/35. \) [[Alternatively: Place one red plate. Then there are \( \binom{4}{1} = 10 \) equally likely ways remaining for the blue plates; in 4 of these ways, the blue plates are adjacent (and therefore the red plates are adjacent too), so the probability is \( 4/35. \)]]

Question 1 was like question \#3 on the 9/15/2014 problem set, with some small changes.

Question 2 was like question \#5 on the 9/29/2014 problem set, with some small changes.

Question 3 was exactly question \#5 on the “more practice problems” for October 5, with the variance included; it was also like question \#1 on the 10/5/2015 in-class problem set.

Question 4 was like question \#3 on the 10/2/2015 in-class set, with some small changes.

Question 5 was exactly like question \#3 on the 8/29/2014 problem set, with 2 and 3 changed to 3 and 4, respectively.