STAT/MA 41600
In-Class Problem Set #9: September 12, 2016
Solutions by Mark Daniel Ward

Problem Set 9 Answers

1. We have $P(X = Y) = p_{X,Y}(1,1) + p_{X,Y}(2,2) + p_{X,Y}(3,3) + p_{X,Y}(4,4) = 1/24 + 1/24 + 1/24 + 1/24 = 1/6$.

2a. The random variables X and Y are not independent. For instance, if $X = 5$ then $Y = 3$ is possible, i.e., $P(Y = 3 \mid X = 5) > 0$, but if $X = 2$ then $Y = 3$ is impossible, i.e., $P(Y = 3 \mid X = 2) = 0$. So the value of X affects the distribution of Y.

2b. If $X = 4$, then Y is one of the values 3, 4, 5, or 6, and all four of these values are equally likely. Thus, we have $p_{Y|X}(1 \mid 4) = 0$; $p_{Y|X}(2 \mid 4) = 0$; $p_{Y|X}(3 \mid 4) = 1/4$; $p_{Y|X}(4 \mid 4) = 1/4$; $p_{Y|X}(5 \mid 4) = 1/4$; $p_{Y|X}(6 \mid 4) = 1/4$.

3a. We compute $P(X = Y) = \sum_{n=1}^{\infty} p_{X,Y}(n,n) = \sum_{n=1}^{\infty}(1/3)(2/3)^{n-1}(3/4)(1/4)^{n-1} = (1/3)(3/4)\sum_{n=1}^{\infty}(1/6)^{n-1} = (1/4)/(1-1/6) = (1/4)/(5/6) = 3/10$.

3b. We compute

$$\begin{aligned}
P(X > Y) &= \sum_{y=1}^{\infty}\sum_{x=y+1}^{\infty} p_{X,Y}(x,y) \\
&= \sum_{y=1}^{\infty}\sum_{x=y+1}^{\infty} (1/3)(2/3)^{x-1}(3/4)(1/4)^{y-1} \\
&= (1/3)(3/4)\sum_{y=1}^{\infty}(1/4)^{y-1}\sum_{x=y+1}^{\infty}(2/3)^{x-1} \\
&= (1/4)\sum_{y=1}^{\infty}(1/4)^{y-1}(2/3)^{y}/(1-2/3) \\
&= (1/2)\sum_{y=1}^{\infty}(1/6)^{y-1} \\
&= (1/2)/(1-1/6) \\
&= (1/2)/(5/6) \\
&= 3/5
\end{aligned}$$

4a. We need to calculate $p_{Y|X}(y|2)$. We have $p_{Y|X}(y|2) = P(Y = y \mid X = 2) = \frac{P(Y=y\ \&\ X=2)}{P(X=2)}$. Thus, for $y \geq 2$, we have

$$\begin{aligned} p_{Y|X}(y|2) &= \frac{P(Y = y\ \&\ X = 2)}{P(X = 2)} \\ &= \frac{(5/9)(1/2)^{2-1}(1/3)^{y-1}}{\sum_{y=2}^{\infty}(5/9)(1/2)^{2-1}(1/3)^{y-1}} \\ &= \frac{(1/3)^{y-1}}{\sum_{y=2}^{\infty}(1/3)^{y-1}} \\ &= \frac{(1/3)^{y-1}}{(1/3)^{2-1}/(1-1/3)} \\ &= \frac{(1/3)^{y-1}}{(1/3)/(2/3)} \\ &= (2)(1/3)^{y-1} \end{aligned}$$

Thus $P(Y > 5 \mid X = 2) = \sum_{y=6}^{\infty} p_{Y|X}(y|2) = \sum_{y=6}^{\infty}(2)(1/3)^{y-1} = (2)(1/3)^5/(1-1/3) = (2)(1/3)^5/(1-1/3) = 1/81$.

4b. The random variables X and Y are dependent. For instance, if $X = 5$ then $Y = 7$ is possible, i.e., $P(Y = 7 \mid X = 5) > 0$, but if $X = 15$ then $Y = 7$ is impossible, i.e., $P(Y = 7 \mid X = 15) = 0$. So the value of X affects the distribution of Y.

4c. The probability mass function of X for $x \geq 1$ is

$$\begin{aligned} p_X(x) &= P(X = x) \\ &= \sum_{y=x}^{\infty} P(X = x\ \&\ Y = y) \\ &= \sum_{y=x}^{\infty}(5/9)(1/2)^{x-1}(1/3)^{y-1} \\ &= (5/9)(1/2)^{x-1}\sum_{y=x}^{\infty}(1/3)^{y-1} \\ &= (5/9)(1/2)^{x-1}(1/3)^{x-1}/(1-1/3) \\ &= (5/6)(1/6)^{x-1} \end{aligned}$$

and $p_X(x) = 0$ otherwise.