Problem Set 9 Answers

1. We have \( P(X = Y) = p_{X,Y}(1, 1) + p_{X,Y}(2, 2) + p_{X,Y}(3, 3) + p_{X,Y}(4, 4) = 1/24 + 1/24 + 1/24 + 1/24 = 1/6. \)

2a. The random variables \( X \) and \( Y \) are not independent. For instance, if \( X = 5 \) then \( Y = 3 \) is possible, i.e., \( P(Y = 3 \mid X = 5) > 0 \), but if \( X = 2 \) then \( Y = 3 \) is impossible, i.e., \( P(Y = 3 \mid X = 2) = 0 \). So the value of \( X \) affects the distribution of \( Y \).

2b. If \( X = 4 \), then \( Y \) is one of the values 3, 4, 5, or 6, and all four of these values are equally likely. Thus, we have
\[
\begin{align*}
p_{Y \mid X}(1 \mid 4) &= 0; \\
p_{Y \mid X}(2 \mid 4) &= 0; \\
p_{Y \mid X}(3 \mid 4) &= 1/4; \\
p_{Y \mid X}(4 \mid 4) &= 1/4; \\
p_{Y \mid X}(5 \mid 4) &= 1/4; \\
p_{Y \mid X}(6 \mid 4) &= 1/4.
\end{align*}
\]

3a. We compute \( P(X = Y) = \sum_{n=1}^{\infty} p_{X,Y}(n, n) = \sum_{n=1}^{\infty} (1/3)(2/3)^{n-1}(3/4)(1/4)^{n-1} = (1/3)(3/4)\sum_{n=1}^{\infty}(1/6)^{n-1} = (1/4)/(1 - 1/6) = (1/4)/(5/6) = 3/10. \)

3b. We compute
\[
P(X > Y) = \sum_{y=1}^{\infty} \sum_{x=y+1}^{\infty} p_{X,Y}(x,y)
\]
\[
= \sum_{y=1}^{\infty} \sum_{y=1}^{\infty} (1/3)(2/3)^{x-1}(3/4)(1/4)^{y-1}
\]
\[
= (1/3)(3/4)\sum_{y=1}^{\infty}(1/4)^{y-1} \sum_{x=y+1}^{\infty}(2/3)^{x-1}
\]
\[
= (1/4)\sum_{y=1}^{\infty}(1/4)^{y-1}(2/3)^{y}/(1 - 2/3)
\]
\[
= (1/2)\sum_{y=1}^{\infty}(1/6)^{y-1}
\]
\[
= (1/2)/(1 - 1/6)
\]
\[
= (1/2)/(5/6)
\]
\[
= 3/5
\]
4a. We need to calculate $p_{Y|X}(y|2)$. We have $p_{Y|X}(y|2) = P(Y = y \mid X = 2) = \frac{P(Y=y \& X=2)}{P(X=2)}$. Thus, for $y \geq 2$, we have

$$p_{Y|X}(y|2) = \frac{P(Y = y \& X = 2)}{P(X = 2)}$$
$$= \frac{(5/9)(1/2)^{y-1}(1/3)^{y-1}}{\sum_{y=2}^{\infty}(5/9)(1/2)^{y-1}(1/3)^{y-1}}$$
$$= \frac{(1/3)^{y-1}}{\sum_{y=2}^{\infty}(1/3)^{y-1}}$$
$$= \frac{(1/3)^{y-1}}{(1/3)^{2-1}/(1 - 1/3)}$$
$$= \frac{(1/3)^{y-1}}{(1/3)/(2/3)}$$
$$= (2)(1/3)^{y-1}$$

Thus $P(Y > 5 \mid X = 2) = \sum_{y=6}^{\infty} p_{Y|X}(y|2) = \sum_{y=6}^{\infty} (2)(1/3)^{y-1} = (2)(1/3)^5/(1 - 1/3) = (2)(1/3)^5/(1 - 1/3) = 1/81$.

4b. The random variables $X$ and $Y$ are dependent. For instance, if $X = 5$ then $Y = 7$ is possible, i.e., $P(Y = 7 \mid X = 5) > 0$, but if $X = 15$ then $Y = 7$ is impossible, i.e., $P(Y = 7 \mid X = 15) = 0$. So the value of $X$ affects the distribution of $Y$.

4c. The probability mass function of $X$ for $x \geq 1$ is

$$p_X(x) = P(X = x)$$
$$= \sum_{y=x}^{\infty} P(X = x \& Y = y)$$
$$= \sum_{y=x}^{\infty} (5/9)(1/2)^{x-1}(1/3)^{y-1}$$
$$= (5/9)(1/2)^{x-1}\sum_{y=x}^{\infty}(1/3)^{y-1}$$
$$= (5/9)(1/2)^{x-1}(1/3)^{x-1}/(1 - 1/3)$$
$$= (5/6)(1/6)^{x-1}$$

and $p_X(x) = 0$ otherwise.