STAT/MA 41600
In-Class Problem Set #11: September 16, 2016
Solutions by Mark Daniel Ward

Problem Set 11 Answers

1. Let X_1, X_2, X_3 indicate (respectively) whether the red, green, and blue pairs are sitting together. For instance, let $X_2 = 1$ if the green pair sits together, and $X_2 = 0$ otherwise. Then $X_j = 1$ with probability $2/5$, so $\mathbb{E}(X_j) = 2/5$ for each j. So we conclude that $\mathbb{E}(X) = \mathbb{E}(X_1 + X_2 + X_3) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = 2/5 + 2/5 + 2/5 = 6/5$.

Another possibility is to number the six chairs, and to let X_j indicate if the jth chair and the chair to its right-hand-side contain a matching color of bear. Then $X_j = 1$ with probability $1/5$, so $\mathbb{E}(X_j) = 1/5$ for each j. So we conclude that $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_6) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_6) = 1/5 + \cdots + 1/5 = 6/5$.

2a. Let X_1, X_2 indicate (respectively) whether the cards drawn in your left and right hand has a number on it. For instance, let $X_2 = 1$ if the card in your right hand has a number on it, and $X_2 = 0$ otherwise. Then $X_j = 1$ with probability $36/52$, so $\mathbb{E}(X_j) = 36/52$ for each j. So we conclude that $\mathbb{E}(X) = \mathbb{E}(X_1 + X_2) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 36/52 + 36/52 = 72/52 = 18/13$.

Another possibility is to consecutively number the 36 cards that have values 2 through 10 in each of the 4 suits, and to let X_j indicate if the jth card is chosen. Then $X_j = 1$ with probability $2/52$, so $\mathbb{E}(X_j) = 2/52$ for each j. So we conclude that $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_{36}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{36}) = 2/52 + \cdots + 2/52 = (36)(2/52) = 72/52 = 18/13$.

2b. Let X_1, X_2, X_3 indicate (respectively) whether the cards drawn in your left hand, right hand, or third hand (you need three hands for this method—i.e., you need a friend to help you with this) has a number on it. For instance, let $X_2 = 1$ if the card in your right hand has a number on it, and $X_2 = 0$ otherwise. Then $X_j = 1$ with probability $36/52$, so $\mathbb{E}(X_j) = 36/52$ for each j. So we conclude that $\mathbb{E}(X) = \mathbb{E}(X_1 + X_2 + X_3) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = 36/52 + 36/52 + 36/52 = 108/52 = 27/13$.

Another possibility is to consecutively number the 36 cards that have values 2 through 10 in each of the 4 suits, and to let X_j indicate if the jth card is chosen. Then $X_j = 1$ with probability $3/52$, so $\mathbb{E}(X_j) = 3/52$ for each j. So we conclude that $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_{36}) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{36}) = 3/52 + \cdots + 3/52 = (36)(3/52) = 108/52 = 27/13$.

3a. Let X_j indicate whether j or more draws are needed. For instance, let $X_8 = 1$ if 8 or more draws are needed, and $X_8 = 0$ otherwise. Then $X_j = 1$ with probability $(4/5)^{j-1}$, so $\mathbb{E}(X_j) = (4/5)^{j-1}$ for each j. So we conclude that $\mathbb{E}(X) = \sum_{j=1}^{\infty}(4/5)^{j-1} = 1/(1-4/5) = 5$.

3b. Let X_j indicate whether j or more draws are needed. For instance, let $X_2 = 1$ if 2 or more draws are needed, and $X_2 = 0$ otherwise. Then $X_1 = 1$ with probability 1, and $X_2 = 1$ with probability $4/5$, and $X_3 = 1$ with probability $(4/5)(3/4) = 3/5$, and $X_4 = 1$ with probability $(4/5)(3/4)(2/3) = 2/5$, and $X_5 = 1$ with probability $(4/5)(3/4)(2/3)(1/2) = 1/5$, so $\mathbb{E}(X_1) = 1$ and $\mathbb{E}(X_2) = 4/5$ and $\mathbb{E}(X_3) = 3/5$ and $\mathbb{E}(X_4) = 2/5$ and $\mathbb{E}(X_5) = 1/5$ and we conclude that $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_5) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_5) = 1 + 4/5 + 3/5 + 2/5 + 1/5 = 3$.

Another possibility is to number the 4 cards that do not have value "A", and to let X_j indicate if the jth card is chosen before the "A". Then $X_j = 1$ with probability $1/2$, so $\mathbb{E}(X_j) = 1/2$ for each j. We know that $X = 1 + X_1 + X_2 + X_3 + X_4$ (the "1" is there because the draw of “A” always counts as 1 draw). So we conclude $\mathbb{E}(X) = \mathbb{E}(1 + X_1 + X_2 + X_3 + X_4) = 1 + \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) + \mathbb{E}(X_4) = 1 + 1/2 + 1/2 + 1/2 + 1/2 = 3$.

4. Let X_j indicate whether the maximum is bigger than or equal to j. So we have $X_1 = 1$ and $X_2 = 1-(1/4)(1/6) = 23/24$ and $X_3 = 1-(2/4)(2/6) = 5/6$ and $X_4 = 1-(3/4)(3/6) = 5/8$ and $X_5 = 1 - (1)(4/6) = 1/3$ and $X_6 = 1 - (1)(5/6) = 1/6$, So we conclude that $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_6) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_6) = 1 + 23/24 + 5/6 + 5/8 + 1/3 + 1/6 = 47/12$.