STAT/MA 41600
In-Class Problem Set #15: September 21, 2016
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Problem Set 15 Answers

1. We have

$$\begin{aligned}
P(|X-Y|=1) &= p_{X,Y}(0,1)+p_{X,Y}(1,2)+p_{X,Y}(2,3)+p_{X,Y}(3,4)\\
&\quad + p_{X,Y}(1,0)+p_{X,Y}(2,1)+p_{X,Y}(3,2)+p_{X,Y}(4,3)\\
&= 2(p_{X,Y}(0,1)+p_{X,Y}(1,2)+p_{X,Y}(2,3)+p_{X,Y}(3,4))\\
&= 2\bigg[\binom{4}{0}(2/5)^0(3/5)^4\binom{4}{1}(2/5)^1(3/5)^3+\binom{4}{1}(2/5)^1(3/5)^3\binom{4}{2}(2/5)^2(3/5)^2\\
&\quad +\binom{4}{2}(2/5)^2(3/5)^2\binom{4}{3}(2/5)^3(3/5)^1+\binom{4}{3}(2/5)^3(3/5)^1\binom{4}{4}(2/5)^4(3/5)^0\bigg]\\
&= \frac{2}{5^8}\bigg[(2)^0(3)^4(4)(2)^1(3)^3+(4)(2)^1(3)^3(6)(2)^2(3)^2\\
&\quad +(6)(2)^2(3)^2(4)(2)^3(3)^1+(4)(2)^3(3)^1(2)^4(3)^0\bigg]\\
&= 172848/390625 = 0.4425
\end{aligned}$$

2a. We have $P(X\le 2)=P(X=0)+P(X=1)+P(X=2)=\binom{5}{0}(1/2)^5+\binom{5}{1}(1/2)^5+\binom{5}{2}(1/2)^5=1/2$. (We could have also realized that, by the symmetry coming from $p=1/2$, it must be the case that $P(X\le 2)$ and $P(X\ge 3)$ are equal, so it follows that $P(X\le 2)=1/2$.)
2b. We have

$$\begin{aligned}
P(X=Y) &= P(X=Y=0)+P(X=Y=1)+P(X=Y=2)\\
&\quad +P(X=Y=3)+P(X=Y=4)+P(X=Y=5)\\
&= \left(\binom{5}{0}(1/2)^5\right)^2+\left(\binom{5}{1}(1/2)^5\right)^2+\left(\binom{5}{2}(1/2)^5\right)^2\\
&\quad +\left(\binom{5}{3}(1/2)^5\right)^2+\left(\binom{5}{4}(1/2)^5\right)^2+\left(\binom{5}{5}(1/2)^5\right)^2\\
&= 2\left[\left(\binom{5}{0}(1/2)^5\right)^2+\left(\binom{5}{1}(1/2)^5\right)^2+\left(\binom{5}{2}(1/2)^5\right)^2\right]\\
&= \frac{2}{1024}(1+25+100) = 252/1024 = 63/256
\end{aligned}$$

We know $P(X>Y)+P(X=Y)+P(X<Y)=1$, which becomes $2P(X>Y)+63/256=1$, so $P(X>Y)=193/512$. Thus $P(X\ge Y)=P(X=Y)+P(X>Y)=63/256+193/512=319/512=0.6230$.
2c. Yes, $X+Y$ is a Binomial random variable with $n=10$ and $p=1/2$.
2d. No, $X-Y$ is not a Binomial random variable. For instance, $X-Y$ can take on negative values, which is not possible for Binomial random variables.

3a. We have $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y) = (5)(1/2) + (5)(1/2) = 5$.
3b. We have $\mathbb{E}(X-Y) = \mathbb{E}(X) - \mathbb{E}(Y) = (5)(1/2) - (5)(1/2) = 0$.
3c. We have $\operatorname{Var}(X+Y) = \operatorname{Var} X + \operatorname{Var} Y = (5)(1/2)(1/2) + (5)(1/2)(1/2) = 5/2$.
3d. We have $\operatorname{Var}(X-Y) = \operatorname{Var} X + (-1)^2 \operatorname{Var} Y = (5)(1/2)(1/2) + (5)(1/2)(1/2) = 5/2$.

4a. The probability that X is even is $P(X=0)+P(X=2)+P(X=4) = \binom{5}{0}(1/3)^0(2/3)^5 + \binom{5}{2}(1/3)^2(2/3)^3 + \binom{5}{4}(1/3)^4(2/3)^1 = \frac{1}{3^5}(32+80+10) = 122/243 = 0.5021$.
4b. The probability that X and Y are equal is $P(X=Y=0) + P(X=Y=1) + P(X=Y=2) + P(X=Y=3) + P(X=Y=4) + P(X=Y=5)$ which evaluates to

$$\begin{aligned} &\binom{5}{0}(1/3)^0(2/3)^5\binom{5}{0}(1/2)^0(1/2)^5 + \binom{5}{1}(1/3)^1(2/3)^4\binom{5}{1}(1/2)^1(1/2)^4 \\ &+ \binom{5}{2}(1/3)^2(2/3)^3\binom{5}{2}(1/2)^2(1/2)^3 + \binom{5}{3}(1/3)^3(2/3)^2\binom{5}{3}(1/2)^3(1/2)^2 \\ &+ \binom{5}{4}(1/3)^4(2/3)^1\binom{5}{4}(1/2)^4(1/2)^1 + \binom{5}{5}(1/3)^5(2/3)^0\binom{5}{5}(1/2)^5(1/2)^0 \end{aligned}$$

which evaluates to $\frac{1}{(2^5)(3^5)}(32+400+800+400+50+1) = 187/864 = 0.2164$.