STAT/MA 41600
In-Class Problem Set #26: October 17, 2016
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Problem Set 26 Answers

1a. Yes, the joint density of X and Y can be factored, namely, as $f_X(x) = 5e^{-5x}$ for $x > 0$, and $f_X(x) = 0$ otherwise, and also $f_Y(y) = 3e^{-3y}$ for $y > 0$, and $f_Y(y) = 0$ otherwise.
1b. For $a > 0$, we compute $P(a \leq Z) = P(a \leq \min(X, Y)) = \int_a^\infty \int_a^\infty 15e^{-5x-3y}\,dy\,dx = \int_a^\infty -5e^{-5x-3y}|_{y=a}^\infty\,dx = \int_a^\infty 5e^{-5x-3a}\,dx = -e^{-5x-a}|_{x=a}^\infty = e^{-8a}$. Thus, for $z > 0$, we have $F_Z(z) = P(Z \leq z) = 1 - P(z \leq Z) = 1 - e^{-8z}$, so we get $f_Z(z) = \frac{d}{dz}(1 - e^{-8z}) = 8e^{-8z}$ for $z > 0$, and $f_Z(z) = 0$ otherwise.

2a. The random variables X and Y are not independent. Perhaps the easiest way to observe this is to note that X and Y are defined in a triangular region of the plane, rather than in rectangular region(s).
2b. We have $P(Y > 2X) = \int_0^\infty \int_{2x}^\infty 24e^{-5x-3y}\,dy\,dx = \int_0^\infty -8e^{-5x-3y}|_{y=2x}^\infty\,dx = \int_0^\infty 8e^{-11x}\,dx = -(8/11)e^{-11x}|_{x=0}^\infty = 8/11$.
3. We compute $P(X > 1/10) = \int_{1/10}^\infty \int_x^\infty 24e^{-5x-3y}\,dy\,dx = \int_{1/10}^\infty -8e^{-5x-3y}|_{y=x}^\infty\,dx = \int_{1/10}^\infty 8e^{-8x}\,dx = -e^{-8x}|_{x=1/10}^\infty = e^{-8/10}$.

4a. The random variables X and Y are not independent because $f_{X,Y}(x, y)$ cannot be factored into an expression in x times an expression in y.
4b. We compute $f_X(x) = \int_0^2 \frac{1}{12}(4 - xy)\,dy = \frac{1}{12}(4y - xy^2/2)|_{y=0}^2 = \frac{1}{12}(8 - 2x) = 2/3 - x/6$ for $0 < x < 2$, and $f_X(x) = 0$ otherwise.
4c. Yes, we have $f_X(x) \geq 0$ for all x, and also $\int_{-\infty}^\infty f_X(x)\,dx = \int_0^2 (2/3 - x/6)\,dx = 1$.