

Problem Set 27 Answers

1a. Yes, it is true, since X and Y are independent.

1b. Since X and Y are independent, we have $P(Y > 3/10 | X > 1/10) = P(Y > 3/10)$. As observed in 1a of the solutions to the previous problem set, we have $f_Y(y) = 3e^{-3y}$ for $y > 0$, and $f_Y(y) = 0$ otherwise. Thus $P(Y > 3/10) = \int_{3/10}^{\infty} 3e^{-3y} dy = -e^{-3y} \Big|_{y=3/10}^{\infty} = e^{-9/10}$.

2a. We have $P(Y > 3/10 | X > 1/10) = \frac{P(Y > 3/10 \ \& \ X > 1/10)}{P(X > 1/10)} = \frac{P(Y > 3/10 \ \& \ X > 1/10)}{e^{-8/10}}$. We compute $P(Y > 3/10 \ \& \ X > 1/10) = \int_{3/10}^{\infty} \int_{1/10}^y 24e^{-5x-3y} dx dy = \int_{3/10}^{\infty} -(24/5)e^{-5x-3y} \Big|_{x=1/10}^y dy = \int_{3/10}^{\infty} ((24/5)e^{-1/2-3y} - (24/5)e^{-8y}) dy = (-8/5)e^{-1/2-3y} + (3/5)e^{-8y} \Big|_{y=3/10}^{\infty} = (8/5)e^{-1/2-3(3/10)} - (3/5)e^{-8(3/10)} = (8/5)e^{-14/10} - (3/5)e^{-24/10}$. So altogether we get $P(Y > 3/10 | X > 1/10) = \frac{(8/5)e^{-14/10} - (3/5)e^{-24/10}}{e^{-8/10}} = (8/5)e^{-6/10} - (3/5)e^{-16/10}$.

2b. We have $f_X(x) = \int_x^{\infty} 24e^{-5x-3y} dy = -8e^{-5x-3y} \Big|_{y=x}^{\infty} = 8e^{-8x}$ for $x > 0$, and $f_X(x) = 0$ otherwise, so $f_{Y|X}(y | x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{24e^{-5x-3y}}{8e^{-8x}} = 3e^{3x-3y}$ for $y > x$, and $f_{Y|X}(y | x) = 0$ otherwise.

2c. We have $P(Y > 3/10 | X = 1/10) = \int_{3/10}^{\infty} f_{Y|X}(y | 1/10) dy = \int_{3/10}^{\infty} 3e^{3/10-3y} dy = e^{3/10-3y} \Big|_{y=3/10}^{\infty} = e^{-6/10}$

3a. Since the joint pdf of X and Y is constant on a triangle with area 16, then $f_{X,Y}(x,y) = 1/16$ for X, Y in the triangle, and $f_{X,Y}(x,y) = 0$ otherwise. Also $f_X(x) = \int_0^{4-x/2} 1/16 dy = (1/16)(4-x/2)$. Thus, we have $f_{Y|X}(y | x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1/16}{(1/16)(4-x/2)} = 1/(4-x/2)$, and in particular, $f_{Y|X}(y | x) = 1/3$.

3b. We have $P(Y > 1 | X = 2) = \int_1^3 f_{Y|X}(y | 2) dy = \int_1^3 1/3 dy = 2/3$.

3c. We have $P(Y > 1 | X > 2) = \frac{P(Y > 1 \ \& \ X > 2)}{P(X > 2)} = \frac{\int_1^3 \int_2^6 1/16 dx dy}{\int_0^3 \int_2^8 1/16 dx dy} = \frac{(1/16)(4)(2)/2}{(1/16)(6)(3)/2} = 4/9$.

4a. We have $f_{Y|X}(y | x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{(1/12)(4-xy)}{2/3-x/6}$ for $0 < x < 2$ and $0 < y < 2$.

4b. We have $f_{Y|X}(y | 2/3) = \frac{f_{X,Y}(2/3,y)}{f_X(2/3)} = \frac{(1/12)(4-(2/3)y)}{2/3-(2/3)/6} = (3/20)(4-(2/3)y)$ for $0 < y < 2$, and $f_{Y|X}(y | 2/3) = 0$ otherwise. Therefore, we compute $P(Y < 4/3 | X = 2/3) = \int_0^{4/3} (3/20)(4-(2/3)y) dy = (3/20)(4y - (2/3)y^2/2) \Big|_{y=0}^{4/3} = 32/45$

4c. We compute $P(Y < 4/3 | X < 2/3) = \frac{P(Y < 4/3 \ \& \ X < 2/3)}{P(X < 2/3)} = \frac{\int_0^{4/3} \int_0^{2/3} (1/12)(4-xy) dx dy}{\int_0^2 \int_0^{2/3} (1/12)(4-xy) dx dy} = \frac{\int_0^{4/3} (1/12)(4x-yx^2/2) \Big|_{x=0}^{2/3} dy}{\int_0^2 (1/12)(4x-yx^2/2) \Big|_{x=0}^{2/3} dy} = \frac{\int_0^{4/3} (1/12)(8/3-4y/18) dy}{\int_0^2 (1/12)(8/3-4y/18) dy} = \frac{(1/12)((8/3)y-4y^2/36) \Big|_{y=0}^{4/3}}{(1/12)((8/3)y-4y^2/36) \Big|_{y=0}^2} = \frac{(1/12)((8/3)(4/3)-4(4/3)^2/36)}{(1/12)((8/3)(2)-4(2)^2/36)} = 68/99$.