STAT/MA 41600
In-Class Problem Set #28: October 21, 2016
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Problem Set 28 Answers

1a. We have noted that $f_{X,Y}(x,y)$ can be factored in this problem, so that $f_X(x) = 5e^{-5x}$ for $x > 0$ and $f_X(x) = 0$ otherwise, and $f_Y(y) = 3e^{-3y}$ for $y > 0$ and $f_Y(y) = 0$ otherwise. So $\mathbb{E}(X) = \int_0^\infty (x)(5e^{-5x})\,dx$. Using integration by parts with $u = x$ and $dv = 5e^{-5x}\,dx$, we have $du = dx$ and $v = -e^{-5x}$, so $\mathbb{E}(X) = (x)(-e^{-5x})|_{x=0}^{\infty} - \int_0^\infty -e^{-5x}\,dx = \int_0^\infty e^{-5x}\,dx = -(1/5)e^{-5x}|_{x=0}^{\infty} = 1/5$.

Alternatively, we could have computed $\mathbb{E}(X) = \int_0^\infty\int_0^\infty (x)(15e^{-5x-3y})\,dx\,dy$, and we would get the exact same answer.

1b. Using the same steps as in 1a, we compute $\mathbb{E}(Y) = \int_0^\infty (y)(3e^{-3y})\,dy = 1/3$.

2a. We compute $\mathbb{E}(X) = \int_0^\infty\int_x^\infty (x)(24e^{-5x-3y})\,dy\,dx = \int_0^\infty (x)(24e^{-5x})\int_x^\infty (e^{-3y})\,dy\,dx = \int_0^\infty (x)(24e^{-5x})(-1/3)(e^{-3y})|_{y=x}^{\infty}\,dx = \int_0^\infty (x)(24e^{-5x})(1/3)(e^{-3x})\,dx = \int_0^\infty (x)(8e^{-8x})\,dx$. The rest of the computation just looks like question 1a, with 8 instead of 5. We have $\mathbb{E}(X) = (x)(-e^{-8x})|_{x=0}^{\infty} - \int_0^\infty -e^{-8x}\,dx = \int_0^\infty e^{-8x}\,dx = -(1/8)e^{-8x}|_{x=0}^{\infty} = 1/8$.

2b. We compute $\mathbb{E}(Y) = \int_0^\infty\int_x^\infty (y)(24e^{-5x-3y})\,dy\,dx = \int_0^\infty (24e^{-5x})\int_x^\infty (ye^{-3y})\,dy\,dx$. For the inner integral, namely, $\int_x^\infty (ye^{-3y})\,dy$, we use integration by parts, with $u = y$ and $dv = e^{-3y}\,dy$, and thus $du = dy$ and $v = -e^{-3y}/3$, so we have $\int_x^\infty (ye^{-3y})\,dy = (y)(-e^{-3y}/3)|_{y=x}^{\infty} - \int_x^\infty -e^{-3y}/3\,dy = (x)(e^{-3x}/3) + (1/9)e^{-3x} = (x/3 + 1/9)e^{-3x}$. Substituting back into the expression for $\mathbb{E}(Y)$, we get $\mathbb{E}(Y) = \int_0^\infty (24e^{-5x})(x/3 + 1/9)e^{-3x}\,dx = \int_0^\infty (8x + 8/3)e^{-8x}\,dx$. We use integration by parts again, this time with $u = 8x + 8/3$ and $dv = e^{-8x}\,dx$, and thus $du = 8\,dx$ and $v = -e^{-8x}/8$, to get $\mathbb{E}(Y) = (8x + 8/3)(-e^{-8x}/8)|_{x=0}^{\infty} - \int_0^\infty (8)(-e^{-8x}/8)\,dx = (8/3)(1/8) + \int_0^\infty e^{-8x}\,dx = 1/3 + 1/8 = 11/24$.

Alternatively, we could have reversed the order of integration. This gives us $\mathbb{E}(Y) = \int_0^\infty\int_0^y (y)(24e^{-5x-3y})\,dx\,dy = \int_0^\infty (24ye^{-3y})\int_0^y e^{-5x}\,dx\,dy$. Now the inner integral is much easier, just $\int_0^y e^{-5x}\,dx = -e^{-5x}/5|_{x=0}^{y} = (1/5)(1-e^{-5y})$. Substituting back into the expression for $\mathbb{E}(Y)$, we get $\mathbb{E}(Y) = \int_0^\infty (24ye^{-3y})(1/5)(1-e^{-5y})\,dy = (24/5)\int_0^\infty y(e^{-3y} - e^{-8y})\,dy$. We use integration by parts with $u = y$ and $dv = (e^{-3y} - e^{-8y})\,dy$, and thus $du = dy$ and $v = -e^{-3y}/3 + e^{-8y}/8$, to get $\mathbb{E}(Y) = (24/5)(y)(-e^{-3y}/3 + e^{-8y}/8)|_{y=0}^{\infty} - (24/5)\int_0^\infty (-e^{-3y}/3 + e^{-8y}/8)\,dy = (24/5)\int_0^\infty (e^{-3y}/3 - e^{-8y}/8)\,dy = (24/5)(-e^{-3y}/9 + e^{-8y}/64)|_{y=0}^{\infty} = (24/5)(1/9 - 1/64) = 11/24$.

3a. We compute $\mathbb{E}(X) = \int_0^8\int_0^{4-x/2} (x)(1/16)\,dy\,dx = \int_0^8 (4 - x/2)(x)(1/16)\,dy\,dx = (1/16)\int_0^8 (4x - x^2/2)\,dx = (1/16)(2x^2 - x^3/6)|_{x=0}^{8} = (1/16)(128 - 256/3) = 8/3$.

3b. We compute $\mathbb{E}(Y) = \int_0^8\int_0^{4-x/2} (y)(1/16)\,dy\,dx = \int_0^8 (y^2/2)(1/16)|_{y=0}^{4-x/2}\,dx = \int_0^8 (4 - x/2)^2(1/32)\,dx = \int_0^8 (16 - 4x + x^2/4)(1/32)\,dx = (16x - 2x^2 + x^3/12)(1/32)|_{x=0}^{8} = 4/3$.

4. We compute $\mathbb{E}(X) = \int_0^2\int_0^2 (x)(\frac{1}{12}(4 - xy))\,dy\,dx = \int_0^2 (x)(\frac{1}{12}(4y - xy^2/2)|_{y=0}^{2})\,dx = \int_0^2 (x)(\frac{1}{12}(8 - 2x))\,dx = \int_0^2 \frac{1}{12}(8x - 2x^2)\,dx = \frac{1}{12}(4x^2 - 2x^3/3)|_{x=0}^{2} = \frac{1}{12}(16 - 16/3) = 8/9$.