STAT/MA 41600
In-Class Problem Set #32 part 2: October 31, 2016
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Problem Set 32 part 2 Answers

1a. We compute $f_X(x) = \int_x^\infty 15e^{-2x-3y}\,dy = -5e^{-2x-3y}|_{y=x}^\infty = 5e^{-5x}$ for $x > 0$, and $f_X(x) = 0$ otherwise.
1b. We compute $f_Y(y) = \int_0^y 15e^{-2x-3y}\,dx = -(15/2)e^{-2x-3y}|_{x=0}^y = (15/2)(e^{-3y} - e^{-5y})$ for $y > 0$, and $f_Y(y) = 0$ otherwise.

2. We see that $0 < U < 1$, so $-\infty < \ln U < 0$, so $0 < -3\ln U < \infty$. Thus X takes values between 0 and ∞. For $a > 0$, we have $P(X > a) = P(-3\ln U > a) = P(\ln U < -a/3) = P(U < e^{-a/3}) = e^{-a/3}$. Thus, the CDF of X is $F_X(x) = P(X \leq x) = 1 - e^{-x/3}$ for $x > 0$, and $F_X(x) = 0$ otherwise. So X is an exponential random variable with parameter $\lambda = 1/3$, and we conclude that $\mathbb{E}(X) = 3$.

3. One quick way to compute $\max(V, W)$ is to use $V + W = \min(V, W) + \max(V, W)$, so $\mathbb{E}(V + W) = \mathbb{E}(\min(V, W) + \max(V, W))$, which becomes $\mathbb{E}(V) + \mathbb{E}(W) = \mathbb{E}(\min(V, W)) + \mathbb{E}(\max(V, W))$. We know that $\mathbb{E}(V) = 2$ and $\mathbb{E}(W) = 2$. Also, since V and W are independent exponential random variables, the minimum of V and W is an exponential random variable too, and the parameter is the sum of the two parameters of V and W, i.e., $\min(V, W)$ is an exponential random variable with parameter $1/2 + 1/2 = 1$, so $\mathbb{E}(\min(V, W)) = 1$. Therefore, $\mathbb{E}(V) + \mathbb{E}(W) = \mathbb{E}(\min(V, W)) + \mathbb{E}(\max(V, W))$ becomes $2 + 2 = 1 + \mathbb{E}(\max(V, W))$, so we conclude $\mathbb{E}(\max(V, W)) = 3$.

A different way is to write $X = \max(V, W)$. For $a > 0$, we have $F_X(a) = P(X \leq a) = P(\max(V, W) \leq a) = P(V \leq a,\ W \leq a) = P(V \leq a)P(W \leq a) = (1 - e^{-a/2})(1 - e^{-a/2}) = 1 - 2e^{-a/2} + e^{-a}$. Therefore, differentiating, we see that the density of X is $f_X(x) = e^{-x/2} - e^{-x}$ for $x > 0$, and $f_X(x) = 0$ otherwise. Thus $\mathbb{E}(X) = \int_0^\infty (x)(e^{-x/2} - e^{-x})\,dx$, which we can split into two separate integrals, and we can multiply and divide by 2 in the first integral, to get a form we recognize easily. So we get $\mathbb{E}(X) = 2\int_0^\infty (x)(1/2)(e^{-x/2})\,dx - \int_0^\infty e^{-x}\,dx = (2)(2) - 1 = 3$.

4a. The minimum of two independent exponential random variables is an exponential random variable whose parameter is the sum of the parameters, i.e., 2λ. Thus, the expected lifetime is $1/(2\lambda)$.
4b. Similarly reasoning gives expected lifetime $1/(3\lambda)$.
4c. Similarly reasoning gives expected lifetime $1/(n\lambda)$.