STAT/MA 41600
In-Class Problem Set #34: November 4, 2016
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Problem Set 34 Answers

1a. We compute $P(X < 1/2) = \int_0^{1/2} \frac{\Gamma(3+2)}{\Gamma(3)\Gamma(2)} x^{3-1}(1-x)^{2-1}\,dx = \int_0^{1/2} \frac{24}{(2)(1)} x^2(1-x)\,dx = \int_0^{1/2} 12(x^2 - x^3)\,dx = 5/16$, and since we know $0 \leq X \leq 1$, it follows that $P(X > 1/2) = 1 - 5/16 = 11/16$. So X is more likely to be larger than 1/2.
1b. We have $\mathbb{E}(X) = \int_0^1 (x) \frac{\Gamma(3+2)}{\Gamma(3)\Gamma(2)} x^{3-1}(1-x)^{2-1}\,dx = \int_0^1 12x^3(1-x)\,dx = \int_0^1 12(x^3 - x^4)\,dx = 12(x^4/4 - x^5/5)|_{x=0}^1 = 12(1/4 - 1/5) = 3/5$.

2a. We have $P(X > 1/2 \mid X > 1/4) = \frac{P(X>1/2 \ \&\ X>1/4)}{P(X>1/4)} = \frac{P(X>1/2)}{P(X>1/4)}$. The numerator is 11/16, as we saw in 1a. The denominator is: $P(X > 1/4) = \int_{1/4}^1 \frac{\Gamma(3+2)}{\Gamma(3)\Gamma(2)} x^{3-1}(1-x)^{2-1}\,dx = \int_{1/4}^1 12x^2(1-x)\,dx = 243/256$. So altogether we get $P(X > 1/2 \mid X > 1/4) = \frac{11/16}{243/256} = 176/243 = 0.7243$.
2b. We have $P(|X - 1/2| > 2/5) = P(X > 9/10 \text{ or } X < 1/10) = P(X > 9/10) + P(X < 1/10) = \int_0^{1/10} \frac{\Gamma(3+2)}{\Gamma(3)\Gamma(2)} x^{3-1}(1-x)^{2-1}\,dx + \int_{9/10}^1 \frac{\Gamma(3+2)}{\Gamma(3)\Gamma(2)} x^{3-1}(1-x)^{2-1}\,dx = \int_0^{1/10} 12x^2(1-x)\,dx + \int_{9/10}^1 12x^2(1-x)\,dx = 37/10000 + 523/10000 = 7/125 = 0.056$.

3. We have $P(U < X) = \int_0^1\int_u^1 (1)\frac{\Gamma(3+2)}{\Gamma(3)\Gamma(2)} x^{3-1}(1-x)^{2-1}\,dx\,du = \int_0^1\int_u^1 12x^2(1-x)\,dx\,du = \int_0^1 (3u^4 - 4u^3 + 1)\,du = 3/5$.

Alternatively, we have We have $P(U < X) = \int_0^1\int_0^x (1)\frac{\Gamma(3+2)}{\Gamma(3)\Gamma(2)} x^{3-1}(1-x)^{2-1}\,du\,dx = \int_0^1\int_0^x 12x^2(1-x)\,du\,dx = \int_0^1 12x^3(1-x)\,dx = 3/5$.
4. The probability is $(5/6)(1/6) + (5/6)^3(1/6) + (5/6)^5(1/6) + (5/6)^7(1/6) + \cdots = (5/6)(1/6)(1 + (5/6)^2 + (5/6)^4 + (5/6)^6 + \cdots) = (5/6)(1/6)(1 + 25/36 + (25/36)^2 + (25/36)^3 + \cdots) = \frac{(5/6)(1/6)}{1-25/36} = 5/11$.