STAT/MA 41600

In-Class Problem Set #39: November 18, 2016

1. Consider a collection of 6 bears. There is a pair of red bears consisting of one father bear and one mother bear. There is a similar green bear pair, and a similar blue bear pair. These 6 bears are all placed around a circular table with 6 chairs, and all arrangements are equally likely. A bear pair is happy if it is sitting together. Let X denote the number of happy bear pairs. Note that $X = X_1 + X_2 + X_3$, where the X_j's are **dependent** indicators.
1a. Find $\text{Cov}(X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j)$ for $i \neq j$.
1b. Find $\text{Var}(X_i) = \text{Cov}(X_i, X_i) = \mathbb{E}(X_i X_i) - \mathbb{E}(X_i)\mathbb{E}(X_i)$.
1c. Use your solutions to **1a** and **1b** to compute $\text{Var}(X)$. You can double-check your answer by comparing with Problem Set 12, question **1c**.

2. Pick three cards simultaneously at random from a well-shuffled deck of 52 cards. There are 36 cards which have numbers on them (cards 2 through 10, in each of the 4 suits), and there are 16 cards without numbers on them (A, J, Q, K, in each of the 4 suits). Let X be the number of cards that you get with numbers on them. Note that $X = X_1 + X_2 + X_3$, where the X_j's are **dependent** indicators.
2a. Find $\text{Cov}(X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j)$ for $i \neq j$.
2b. Find $\text{Var}(X_i) = \text{Cov}(X_i, X_i) = \mathbb{E}(X_i X_i) - \mathbb{E}(X_i)\mathbb{E}(X_i)$.
2c. Use your solutions to **2a** and **2b** to compute $\text{Var}(X)$. You can double-check your answer by comparing with Problem Set 12, question **3**.

3. Suppose that X and Y have joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{12}(4 - xy) & \text{if } 0 < x < 2 \text{ and } 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $\text{Cov}(X, Y)$.

Hints: When finding $\mathbb{E}(X)$, you might want to refer to Problem Set 26, question #4b, but there are multiple ways to find $\mathbb{E}(X)$. When finding $\mathbb{E}(Y)$, note that, by symmetry, we have $\mathbb{E}(Y) = \mathbb{E}(X)$.

4 (review). Consider a pair of random variables X and Y with joint probability density function $f_{X,Y}(x,y) = \frac{1}{8}xy$ for x, y in the triangle where $0 < x < 2$ and $0 < y < 2x$, and $f_{X,Y}(x,y) = 0$ otherwise.

If $0 < x < 2$, find the conditional density $f_{Y|X}(y \mid x)$ of Y given $X = x$.