STAT/MA 41600
In-Class Problem Set #39 part 2: November 21, 2016
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Problem Set 39 part 2 Answers

1. We know that $\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$. We proved on Friday that $\mathrm{Cov}(X,Y) = -4/81$. We also proved on Friday that $\mathbb{E}(X) = \int_0^2\int_0^2 (x)(1/12)(4-xy)\,dx\,dy = 8/9$. We now compute $\mathbb{E}(X^2) = \int_0^2\int_0^2 (x^2)(1/12)(4-xy)\,dx\,dy = 10/9$. So we get $\mathrm{Var}(X) = 10/9 - (8/9)^2 = 26/81$. Similarly, we have $\mathrm{Var}(Y) = 26/81$. So we conclude that $\rho(X,Y) = \frac{-4/81}{\sqrt{(26/81)(26/81)}} = -2/13$.

2. Let $X_i = 1$ if the ith child gets 1 bear of each color, or $X_i = 0$ otherwise. So we have $X = X_1 + \cdots + X_{10}$. We have $\mathrm{Var}(X) = \sum_{i=1}^{10} \mathrm{Var}(X_i) + \sum_{i\neq j} \mathrm{Cov}(X_i, X_j)$. We have $\mathrm{Var}(X_i) = \mathbb{E}(X_i^2) - (\mathbb{E}(X_i))^2 = (20/29)(10/28) - ((20/29)(10/28))^2 = 7650/41209 = 0.1856$. For $i \neq j$, we have $\mathrm{Cov}(X_i, X_j) = \mathbb{E}(X_iX_j) - \mathbb{E}(X_i)\mathbb{E}(X_j) = (20/29)(10/28)(18/26)(9/25) - (20/29)(10/28)(20/29)(10/28) = 386/535717 = 0.0007205$. So we conclude that $\mathrm{Var}(X) = 10(7650/41209) + 90(386/535717) = 1029240/535717 = 1.92$.

3. Let $X = X_1 + \cdots + X_5$ where X_j indicates whether Alice's jth choice is a Queen, and let $Y = Y_1 + \cdots + Y_5$ where Y_j indicates whether Bob's jth choice is a Queen. We have $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_5) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_5) = 5(1/13) = 5/13$ and $\mathbb{E}(Y) = 5/13$. Finally $\mathbb{E}(XY) = \mathbb{E}((X_1 + \cdots + X_5)(Y_1 + \cdots + Y_5)) = 25\mathbb{E}(X_iY_j) = 25(4/52)(3/51) = 25/221$. Thus $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 25/221 - (5/13)(5/13) = -100/2873 = -0.0348$.

4a. We compute $\mathbb{E}(XY) = \int_0^2\int_0^{2x} (xy)(\frac{1}{8}xy)\,dy\,dx = 32/9$ and $\mathbb{E}(X) = \int_0^2\int_0^{2x} (x)(\frac{1}{8}xy)\,dy\,dx = 8/5$ and $\mathbb{E}(Y) = \int_0^2\int_0^{2x} (y)(\frac{1}{8}xy)\,dy\,dx = 32/15$. Therefore, we get $\mathrm{Cov}(X,Y) = 32/9 - (8/5)(32/15) = 32/225$.

4b. We compute $\mathbb{E}(X^2) = \int_0^2\int_0^{2x} (x^2)(\frac{1}{8}xy)\,dy\,dx = 8/3$ and $\mathbb{E}(Y^2) = \int_0^2\int_0^{2x} (y^2)(\frac{1}{8}xy)\,dy\,dx = 16/3$. This yields $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 8/3 - (8/5)^2 = 8/75$ and $\mathrm{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 16/3 - (32/15)^2 = 176/225$. So altogether we conclude $\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{32/225}{\sqrt{(8/75)(176/225)}} = \frac{2\sqrt{66}}{33} = 0.4924$.