STAT/MA 41600
In-Class Problem Set #41: November 30, 2016
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Problem Set 41 Answers

1a. Let X be the waiting time. Then $P(X \geq 3.5) \leq \frac{2}{3.5} = 0.57$.
1b. If X is exponential with $\mathbb{E}(X) = 2$, then $P(X \geq 3.5) = e^{-3.5/2} = 0.17$.
1c. If X is exponential with $\mathbb{E}(X) = 2$, then $\text{Var}(X) = 4$.

2. Let X be the number of minutes after 7 PM until your significant other arrives. Then $P(10 < X < 30) = P(|X - 20| < 10) = P(|X - 20| < k\sigma_X)$, where $\sigma_X = 5$ so $k = 2$. Therefore we get $P(10 < X < 30) = P(|X - 20| < 10) = P(|X - 20| < k\sigma_X) \geq \frac{2^2-1}{2^2} = 3/4$, where the inequality holds by the Chebyshev Inequality.

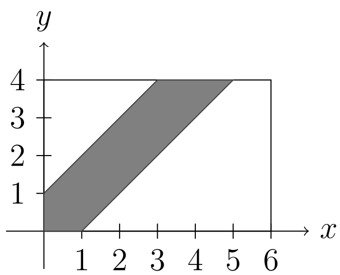
3. Method #1:

We have $P(|X - Y| \leq 3) = P(0 \leq X - Y \leq 3) + P(0 \leq Y - X \leq 3) = P(Y \leq X \leq Y+3) + P(X \leq Y \leq X+3) = \int_0^\infty \int_y^{y+3} (\frac{1}{5}e^{-x/5})(\frac{1}{7}e^{-y/7})\,dx\,dy + \int_0^\infty \int_x^{x+3} (\frac{1}{5}e^{-x/5})(\frac{1}{7}e^{-y/7})\,dy\,dx$, which we see simplifies to $\int_0^\infty (e^{-y/5} - e^{-(y+3)/5})(\frac{1}{7}e^{-y/7})\,dy + \int_0^\infty (\frac{1}{5}e^{-x/5})(e^{-x/7} - e^{-(x+3)/7})\,dx = \frac{1}{7}(1 - e^{-3/5}) \int_0^\infty e^{-12y/35}\,dy + \frac{1}{5}(1 - e^{-3/7}) \int_0^\infty e^{-12x/35}\,dx = \frac{1}{7}(1 - e^{-3/5}) \frac{e^{-12y/35}}{-12/35}\Big|_{y=0}^\infty + \frac{1}{5}(1 - e^{-3/7}) \frac{e^{-12x/35}}{-12/35}\Big|_{x=0}^\infty = \frac{1}{7}(1 - e^{-3/5}) \frac{1}{12/35} + \frac{1}{5}(1 - e^{-3/7}) \frac{1}{12/35} = 1 - \frac{5}{12}e^{-3/5} - \frac{7}{12}e^{-3/7} = 0.3913$.

Method #2:

We have $P(|X - Y| \leq 3) = P(0 \leq X \leq 3 \ \&\ 0 \leq Y \leq X + 3) + P(3 \leq X \ \&\ X - 3 \leq Y \leq X + 3) = \int_0^3 \int_0^{x+3} (\frac{1}{5}e^{-x/5})(\frac{1}{7}e^{-y/7})\,dy\,dx + \int_3^\infty \int_{x-3}^{x+3} (\frac{1}{5}e^{-x/5})(\frac{1}{7}e^{-y/7})\,dy\,dx = 1 - \frac{5}{12}e^{-3/5} - \frac{7}{12}e^{-3/7} = 0.3913$.

4. The shaded region below has area $24 - 3^2/2 - 4^2/2 - 4 = 15/2$, and the entire region has area 24, so the desired probability is $\frac{15/2}{24} = 5/16 = 0.3125$.



Alternatively, we can compute $\int_0^1 \int_0^{y+1} 1/24\,dx\,dy + \int_1^4 \int_{y-1}^{y+1} 1/24\,dx\,dy = 1/16 + 1/4 = 5/16 = 0.3125$.