STAT/MA 41600

In-Class Problem Set #42: December 2, 2016

1. Consider 3 independent random variables X_1, X_2, X_3, each of which has probability density function $x/18$ for $0 < x < 6$, and 0 otherwise. Let $X_{(1)}$, $X_{(2)}$, $X_{(3)}$ be their order statistics.

1a. Find the density of $X_{(1)} = \min(X_1, X_2, X_3)$.

1b. Compute $\mathbb{E}(X_{(1)})$.

1c. Find the density of the second order statistic, $X_{(2)}$, i.e., the second-smallest one.

1d. Compute $\mathbb{E}(X_{(2)})$.

2. Same setup as in **1**.

2a. Find the density of $X_{(3)} = \max(X_1, X_2, X_3)$.

2b. Compute $\mathbb{E}(X_{(3)})$.

2c. Sanity check: We can easily compute $\mathbb{E}(X_1) = \mathbb{E}(X_2) = \mathbb{E}(X_3) = \int_0^6 (x)(x/18)\,dx = 4$ (just trust me; or check it yourself if you want to).

We know that $X_1 + X_2 + X_3 = X_{(1)} + X_{(2)} + X_{(3)}$. Therefore, we have $\mathbb{E}(X_{(1)}) + \mathbb{E}(X_{(2)}) + \mathbb{E}(X_{(3)}) = \mathbb{E}(X_{(1)} + X_{(2)} + X_{(3)}) = \mathbb{E}(X_1 + X_2 + X_3) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = 4 + 4 + 4 = 12$. So please make sure your answers to **1b**, **1d**, and **2b** sum to 12 too.

3. Let $U_1, \ldots, U_7$ be seven independent, continuous random variables, each uniformly distributed on $[0, 5]$.

3a. Find the probability density function of $U_{(3)}$.

3b. Find the mean of $U_{(3)}$, the third-smallest of these seven random variables.

3c. Can you quickly find the mean of $U_{(4)}$ without computing an integral?

4a. Consider a deck of 5 cards with the values A, 2, 3, 4, 5. We deal one card at a time from this deck of 5 cards, with replacement of the card back into the deck—and also shuffling—in between each deal. We continue in this fashion until the first A appears, and then we stop afterwards. Let X be the number of cards dealt. What is the variance of X?

4b. Reconsider question **4a**, but this time do not replace the cards after they are dealt. What is the variance of X?