STAT/MA 41600
In-Class Problem Set #43: December 5, 2016
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Problem Set 43 Answers

1. For $t < 5$, we have $g(t) = M_X(t) = \mathbb{E}(e^{tX}) = \int_0^\infty (e^{tx})(5e^{-5x})\,dx = \int_0^\infty (5e^{(t-5)x})\,dx = \frac{5e^{(t-5)x}}{(t-5)}\Big|_{x=0}^{\infty} = \frac{5}{5-t}$.

2a. We compute $g'(t) = M_X'(t) = \frac{d}{dt}(\frac{5}{5-t}) = \frac{(5-t)(0)-(5)(-1)}{(5-t)^2} = \frac{5}{(5-t)^2}$.
2b. We get $\mathbb{E}(X) = g(0) = \frac{5}{(5-0)^2} = 1/5$.
2c. We already computed the first derivative, which we use here. We have $g''(t) = M_X''(t) = \frac{d^2}{dt^2}(\frac{5}{5-t}) = \frac{d}{dt}(\frac{5}{(5-t)^2}) = \frac{(5-t)^2(0)-(5)(2)(5-t)(-1)}{(5-t)^4} = \frac{10}{(5-t)^3}$. Therefore, we get $\mathbb{E}(X^2) = g''(0) = 10/5^3 = 2/25$.
2d. We conclude that $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 2/25 - (1/5)^2 = 1/25$, which agrees what we know about an exponential random variable with $\lambda = 5$.

3. We have $g(t) = M_X(t) = \mathbb{E}(e^{tX}) = \sum_{x=1}^\infty (e^{tx})(4/5)^{x-1}(1/5) = \frac{1/5}{4/5}\sum_{x=1}^\infty (e^{tx})(4/5)^x = \frac{1/5}{4/5}\sum_{x=1}^\infty (\frac{4}{5}e^t)^x = \frac{1/5}{4/5}(\frac{\frac{4}{5}e^t}{1-\frac{4}{5}e^t}) = \frac{e^t}{5-4e^t}$.

4a. We compute $g'(t) = M_X'(t) = \frac{d}{dt}(\frac{e^t}{5-4e^t}) = \frac{(5-4e^t)(e^t)-(e^t)(-4e^t)}{(5-4e^t)^2} = \frac{5e^t}{(5-4e^t)^2}$.
4b. We get $\mathbb{E}(X) = g(0) = \frac{5e^0}{(5-4e^0)^2} = 5$.
4c. We already computed the first derivative, which we use here. We have $g''(t) = M_X''(t) = \frac{d^2}{dt^2}(\frac{e^t}{5-4e^t}) = \frac{d}{dt}(\frac{5e^t}{(5-4e^t)^2}) = \frac{(5-4e^t)^2(5e^t)-(5e^t)(2)(5-4e^t)(-4e^t)}{(5-4e^t)^4} = \frac{25e^t+20e^{2t}}{(5-4e^t)^3}$. Therefore, we get $\mathbb{E}(X^2) = g''(0) = \frac{25e^0+20e^{2(0)}}{(5-4e^0)^3} = 45$.
4d. We conclude that $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 45 - 5^2 = 20$, which agrees what we know about an geometric random variable with $p = 1/5$ and $q = 4/5$, since the variance should be $q/p^2 = (4/5)^2/(1/5) = 20$.