STAT/MA 41600
In-Class Problem Set #5: September 1, 2017
Solutions by Mark Daniel Ward

Problem Set 5 Answers

1. Let A be the event that she chooses the 52 card deck, and let F be the event that the selected card is a face card. Then $P(A \mid F) = \frac{P(A\cap F)}{P(F)} = \frac{P(A\cap F)}{P(A\cap F)+P(A^c\cap F)} = \frac{P(A)P(F \mid A)}{P(A)P(F \mid A)+P(A^c)P(F \mid A^c)} = \frac{(1/2)(12/52)}{(1/2)(12/52)+(1/2)(12/24)} = 6/19 = 0.3158.$

2. Let M be the event that the student is good at math. Let C be the event that the student is in the College of Science. Then $P(C \mid M) = \frac{P(C\cap M)}{P(M)} = \frac{P(C\cap M)}{P(C\cap M)+P(C^c\cap M)} = \frac{P(C)P(M \mid C)}{P(C)P(M \mid C)+P(C^c)P(M \mid C^c)} = \frac{(0.2)(0.7)}{(0.2)(0.7)+(0.8)(0.6)} = 0.2258.$

3a. Let events A, B, and C be the events (respectively) that coin A, B, or C was chosen. Let H be the event that we get 7 heads when we flip the selected coin. Then $P(A \mid H) = \frac{P(A\cap H)}{P(H)} = \frac{P(A\cap H)}{P(A\cap H)+P(B\cap H)+P(C\cap H)} = \frac{P(A)P(H \mid A)}{P(A)P(H \mid A)+P(B)P(H \mid B)+P(C)P(H \mid C)} = \frac{(\frac{1}{3})(0.49)^7}{(\frac{1}{3})(0.49)^7+(\frac{1}{3})(0.52)^7+(\frac{1}{3})(0.50)^7} = 0.2726.$

3b. We have $P(B \mid H) = \frac{P(B\cap H)}{P(H)} = \frac{P(B\cap H)}{P(A\cap H)+P(B\cap H)+P(C\cap H)} = \frac{P(B)P(H \mid B)}{P(A)P(H \mid A)+P(B)P(H \mid B)+P(C)P(H \mid C)} = \frac{(\frac{1}{3})(0.52)^7}{(\frac{1}{3})(0.49)^7+(\frac{1}{3})(0.52)^7+(\frac{1}{3})(0.50)^7} = 0.4133.$

3c. We have $P(C \mid H) = \frac{P(C\cap H)}{P(H)} = \frac{P(C\cap H)}{P(A\cap H)+P(B\cap H)+P(C\cap H)} = \frac{P(C)P(H \mid C)}{P(A)P(H \mid A)+P(B)P(H \mid B)+P(C)P(H \mid C)} = \frac{(\frac{1}{3})(0.50)^7}{(\frac{1}{3})(0.49)^7+(\frac{1}{3})(0.52)^7+(\frac{1}{3})(0.50)^7} = 0.3141.$

4a. If Alice gets a 1 or 2, then it is impossible for Bob to get at least 3 heads.

If Alice gets a 3, then Bob gets 3 heads with probability $(1/2)^3$.

If Alice gets a 4, then Bob gets 3 heads with probability $\binom{4}{3}(1/2)^3(1/2)^1$, or gets 4 heads with probability $(1/2)^4$.

If Alice gets a 5, then Bob gets 3 heads with probability $\binom{5}{3}(1/2)^3(1/2)^2$, or gets 4 heads with probability $\binom{5}{4}(1/2)^4(1/2)^1$, or gets 5 heads with probability $(1/2)^5$.

If Alice gets a 6, then Bob gets 3 heads with probability $\binom{6}{3}(1/2)^3(1/2)^3$, or gets 4 heads with probability $\binom{6}{4}(1/2)^4(1/2)^2$, or gets 5 heads with probability $\binom{6}{5}(1/2)^5(1/2)^1$. or gets 6 heads with probability $(1/2)^6$.

So the probability that Bob gets at least 3 heads is

$$\begin{aligned}
&(1/6)(0) + (1/6)(0) + (1/6)(1/2)^3 + (1/6)\left(\binom{4}{3}(1/2)^3(1/2)^1 + (1/2)^4\right) \\
&\qquad + (1/6)\left(\binom{5}{3}(1/2)^3(1/2)^2 + \binom{5}{4}(1/2)^4(1/2)^1 + (1/2)^5\right) \\
&\qquad + (1/6)\left(\binom{6}{3}(1/2)^3(1/2)^3 + \binom{6}{4}(1/2)^4(1/2)^2 + \binom{6}{5}(1/2)^5(1/2)^1 + (1/2)^6\right) \\
&= (1/6)(0) + (1/6)(0) + (1/6)(1/8) + (1/6)\big((4)(1/16) + 1/16\big) + (1/6)\big((10)(1/32) + (5)(1/32) + 1/32\big) \\
&\qquad + (1/6)\big((20)(1/64) + (15)(1/64) + (6)(1/64) + 1/64\big) = 17/64
\end{aligned}$$

4b. Let A_j be the event that Alice rolls a j. Let B be the event that Bob gets 2 heads. Then $P(A_4 \mid B) = \frac{P(A_4\cap B)}{P(B)} = \frac{P(A_4\cap B)}{P(A_4\cap B)+P(A_4\cap B)+P(A_4\cap B)+P(A_4\cap B)+P(A_4\cap B)+P(A_4\cap B)}$ which is $\frac{(1/6)\binom{4}{2}(1/2)^2(1/2)^2}{(1/6)(0)+(1/6)(1/2)^2+(1/6)\binom{3}{2}(1/2)^2(1/2)^1+(1/6)\binom{4}{2}(1/2)^2(1/2)^2+(1/6)\binom{5}{2}(1/2)^2(1/2)^3+(1/6)\binom{6}{2}(1/2)^2(1/2)^4} = \frac{8}{33} = 0.2424.$