STAT/MA 41600
In-Class Problem Set #9: September 11, 2017
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Problem Set 9 Answers

1. The probability that Leo takes n rolls and Melissa takes n rolls is $(5/6)^{n-1}(1/6)(1/2)^{n-1}(1/2) = (5/12)^{n-1}(1/12)$. Therefore, we get $P(X = Y) = \sum_{n=1}^{\infty} P(X = Y = n) = \sum_{n=1}^{\infty}(5/12)^{n-1}(1/12) = \frac{1/12}{1-5/12} = 1/7$.

2a. All of the values $p_{X,Y}(x, y)$ are nonnegative, so it suffices to check that they sum to 1:
$p_{X,Y}(1, 1) = 1/7$; $p_{X,Y}(1, 2) = 1/14$; $p_{X,Y}(1, 3) = 1/21$; $p_{X,Y}(1, 4) = 1/28$;
$p_{X,Y}(2, 2) = 1/7$; $p_{X,Y}(2, 3) = 2/21$; $p_{X,Y}(2, 4) = 1/14$;
$p_{X,Y}(3, 3) = 1/7$; $p_{X,Y}(3, 4) = 3/28$;
and $p_{X,Y}(4, 4) = 1/7$; so, yes indeed, the probabilities sum to 1.
2b. We have $p_X(1) = 1/7 + 1/14 + 1/21 + 1/28 = 25/84$; $p_X(2) = 1/7 + 2/21 + 1/14 = 13/42$; $p_X(3) = 1/7 + 3/28 = 1/4$; and $p_X(4) = 1/7$.
2c. We have $p_Y(1) = 1/7$; $p_Y(2) = 1/14 + 1/7 = 3/14$; $p_Y(3) = 1/21 + 2/21 + 1/7 = 2/7$; and $p_Y(4) = 1/28 + 1/14 + 3/28 + 1/7 = 5/14$.

3abcd. We have $p_{X,Y}(0, 0) = (4/6)^3 = 8/27$. We can calculate directly $p_{X,Y}(0, 1) = \binom{3}{1}(1/6)(4/6)^2 + \binom{3}{2}(1/6)^2(4/6)^1 + \binom{3}{3}(1/6)^3 = 61/216$. Alternatively, we can calculate the probability of no 4's minus the probability of no 4's and no 5's, i.e., $p_{X,Y}(0, 1) = (5/6)^3 - (4/6)^3 = 61/216$. Similarly, we have $p_{X,Y}(1, 0) = 61/216$. Finally, to get $p_{X,Y}(1, 1)$, we either have one 4, one 5, and a different value, or we have two 4's and one 5, or we have two 5's and one 4, so $p_{X,Y}(1, 1) = (3!)(1/6)(1/6)(4/6) + \binom{3}{2}(1/6)^2(1/6) + \binom{3}{2}(1/6)^2(1/6) = 5/36$. We can double check that $8/27 + 61/216 + 61/216 + 5/36 = 1$, so we have a valid joint probability mass function.
3e. We have $p_{X|Y}(0|0) = \frac{p_{X,Y}(0,0)}{p_Y(0)} = \frac{p_{X,Y}(0,0)}{p_{X,Y}(0,0)+p_{X,Y}(1,0)} = \frac{8/27}{8/27+61/216} = 64/125$.
Similarly, we have $p_{X|Y}(1|0) = \frac{p_{X,Y}(1,0)}{p_Y(0)} = \frac{p_{X,Y}(1,0)}{p_{X,Y}(0,0)+p_{X,Y}(1,0)} = \frac{61/216}{8/27+61/216} = 61/125$.
3f. We have $p_{X|Y}(0|1) = \frac{p_{X,Y}(0,1)}{p_Y(1)} = \frac{p_{X,Y}(0,1)}{p_{X,Y}(0,1)+p_{X,Y}(1,1)} = \frac{61/216}{61/216+5/36} = 61/91$.
Similarly, we have $p_{X|Y}(1|1) = \frac{p_{X,Y}(1,1)}{p_Y(1)} = \frac{p_{X,Y}(1,1)}{p_{X,Y}(0,1)+p_{X,Y}(1,1)} = \frac{5/36}{61/216+5/36} = 30/91$.

4. We have:
$p_{X|Y}(1|1) = 1$;
$p_{X|Y}(1|2) = \frac{p_{X,Y}(1,2)}{p_Y(2)} = \frac{1/16}{3/16} = 1/3$; and $p_{X|Y}(2|2) = \frac{p_{X,Y}(2,2)}{p_Y(2)} = \frac{2/16}{3/16} = 2/3$;
$p_{X|Y}(1|3) = \frac{p_{X,Y}(1,3)}{p_Y(3)} = \frac{1/16}{5/16} = 1/5$; $p_{X|Y}(2|3) = \frac{p_{X,Y}(2,3)}{p_Y(3)} = \frac{1/16}{5/16} = 1/5$; and $p_{X|Y}(3|3) = \frac{p_{X,Y}(3,3)}{p_Y(3)} = \frac{3/16}{5/16} = 3/5$;
$p_{X|Y}(1|4) = \frac{p_{X,Y}(1,4)}{p_Y(4)} = \frac{1/16}{7/16} = 1/7$; $p_{X|Y}(2|4) = \frac{p_{X,Y}(2,4)}{p_Y(4)} = \frac{1/16}{7/16} = 1/7$; $p_{X|Y}(3|4) = \frac{p_{X,Y}(3,4)}{p_Y(4)} = \frac{1/16}{7/16} = 1/7$; and $p_{X|Y}(4|4) = \frac{p_{X,Y}(4,4)}{p_Y(4)} = \frac{4/16}{7/16} = 4/7$.