STAT/MA 41600
In-Class Problem Set #25: October 13, 2017
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Problem Set 25 Answers

1. We compute as follows: $P(Y > 2X+1) = \int_0^\infty \int_{2x+1}^\infty e^{-x-y}\,dy\,dx = \int_0^\infty -e^{-x-y}|_{y=2x+1}^\infty\,dx = \int_0^\infty e^{-x-(2x+1)}\,dx = \int_0^\infty e^{-3x-1}\,dx = -(1/3)e^{-3x-1}|_{x=0}^\infty = (1/3)(e^{-1})$.

2a. We compute as follows: $P(\max(X,Y) \le 1) = \int_0^1\int_0^1 e^{-x-y}\,dy\,dx = \int_0^1 e^{-x}\,dx \int_0^1 e^{-y}\,dy = (-e^{-x}|_{x=0}^1)(-e^{-y}|_{y=0}^1) = (1-e^{-1})(1-e^{-1}) = (1-e^{-1})^2$.
2b. We compute as follows: $P(\max(X,Y) \le 2) = \int_0^2\int_0^2 e^{-x-y}\,dy\,dx = \int_0^2 e^{-x}\,dx \int_0^2 e^{-y}\,dy = (-e^{-x}|_{x=0}^2)(-e^{-y}|_{y=0}^2) = (1-e^{-2})(1-e^{-2}) = (1-e^{-2})^2$.
2c. For $a > 0$, we compute as follows: $P(\max(X,Y) \le a) = \int_0^a\int_0^a e^{-x-y}\,dy\,dx = \int_0^a e^{-x}\,dx \int_0^a e^{-y}\,dy = (-e^{-x}|_{x=0}^a)(-e^{-y}|_{y=0}^a) = (1-e^{-a})(1-e^{-a}) = (1-e^{-a})^2$.
2d. The CDF of Z is $F_Z(z) = 0$ for $z \le 0$, and $F_Z(z) = (1-e^{-z})^2$ for $z > 0$. So the probability density function of Z is $f_Z(z) = 0$ for $z \le 0$, and $f_Z(z) = \frac{d}{dz}(1-e^{-z})^2 = (2)(1-e^{-z})(e^{-z})$ for $z > 0$.

3a. We have $P(|X-3| < 1/2) = \int_{5/2}^{7/2} f_X(x)\,dx = \int_{5/2}^3 x/9\,dx + \int_3^{7/2}(2/3 - x/9)\,dx = x^2/18|_{x=5/2}^3 + (2x/3 - x^2/18)|_{x=3}^{7/2} = (3^2 - (5/2)^2)/18 + ((2)(7/2)/3 - (7/2)^2/18) - ((2)(3)/3 - 3^2/18) = 11/36$.
3b. We have $P(|X-3| > 2) = \int_0^1 f_X(x)\,dx + \int_5^6 f_X(x)\,dx = \int_0^1 x/9\,dx + \int_5^6 (2/3 - x/9)\,dx = x^2/18|_{x=0}^1 + (2x/3 - x^2/18)|_{x=5}^6 = 1/18 + ((2)(6)/3 - 6^2/18) - ((2)(5)/3 - 5^2/18) = 1/9$.

4a. We have $f_X(x) = 0$ for $x \le 0$. For $x > 0$, we have $f_X(x) = \int_0^\infty f_{X,Y}(x,y)\,dy = \int_0^\infty e^{-x-y}\,dx = -e^{-x-y}|_{y=0}^\infty = e^{-x}$.
4b. For $a > 0$, we have $F_X(a) = \int_0^a f_X(x)\,dx = \int_0^a e^{-x}\,dx = -e^{-x}|_{x=0}^a = 1 - e^{-a}$. Therefore, we get $F_X(x) = 1 - e^{-x}$ for $x > 0$, and $F_X(x) = 0$ otherwise.
4c. For $0 \le a \le 3$, we have $F_X(a) = P(X \le a) = \int_0^a f_X(x)\,dx = \int_0^a x/9\,dx = x^2/18|_{x=0}^a = a^2/18$.

For $3 \le a \le 6$, we have $F_X(a) = P(X \le a) = \int_0^a f_X(x)\,dx = \int_0^3 x/9\,dx + \int_3^a (2/3 - x/9)\,dx = x^2/18|_{x=0}^3 + (2x/3 - x^2/18)|_{x=3}^a = 9/18 + ((2)a/3 - a^2/18) - ((2)(3)/3 - 3^2/18) = -1 + 2a/3 - a^2/18$.

So the CDF of X is:

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^2/18 & \text{for } 0 \le x \le 3 \\ -1 + 2x/3 - x^2/18 & \text{for } 3 \le x \le 6 \\ 1 & \text{for } x > 6 \end{cases}$$