STAT/MA 41600
In-Class Problem Set #31: October 25, 2017
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Problem Set 31 Answers

1. Using the ratio of areas only (since the joint probability density function is constant), we can compute $P(X > 0) = 8/12 = 2/3$.

Alternatively, by integrating, we get $P(X > 0) = \int_0^2 \int_0^{-4y+8} 1/12\,dx\,dy = \int_0^2 (1/12)(-4y + 8)\,dy = (1/12)(-2y^2 + 8y)|_{y=0}^2 = (1/12)(-8 + 16) = 8/12 = 2/3$.

2a. We have $f_X(x) = \frac{d}{dx}F_X(x) = 1/15$ for $5 < x < 20$, and $f_X(x) = 0$ otherwise. So the expected value is $\mathbb{E}(X) = \int_5^{20}(x)(1/15)\,dx = (x^2/2)(1/15)|_{x=5}^{20} = (200 - 25/2)(1/15) = (375/2)(1/15) = (375/2)(1/15) = 25/2$.
2b. We compute $\mathbb{E}(X^2) = \int_5^{20}(x^2)(1/15)\,dx = (x^3/3)(1/15)|_{x=5}^{20} = (8000/3 - 125/3)(1/15) = (2625)(1/15) = 175$. So we conclude $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 175 - (25/2)^2 = 75/4$.

3. If $-3 \le a \le 3$, we have $P(X \le a) = P(U \le a,\ V \le a,\ W \le a) = P(U \le a)P(V \le a)P(W \le a) = ((a+3)/6)^3$. So we have $F_X(x) = ((x+3)/6)^3$ for $-3 \le x \le 3$. Therefore, the probability density function of X is $f_X(x) = \frac{d}{dx}F_X(x) = 3((x+3)/6)^2(1/6) = ((x+3)/6)^2/2$ for $-3 \le x \le 3$, and $f_X(x) = 0$ otherwise.

4. For $0 \le a \le 3$, we have $F_Z(a) = P(Z \le a) = P(X + Y \le a) = (a^2/2)/9$.

For $3 \le a \le 6$, we have $1 - F_Z(a) = 1 - P(Z \le a) = P(Z > a) = P(X + Y > a) = ((6 - a)^2/2)/9$. Thus, we have $F_Z(a) = 1 - ((6 - a)^2/2)/9$.

So we conclude that:
For $0 \le z \le 3$, we have $f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{d}{dz}(z^2/2)/9 = z/9$.
For $3 \le z \le 6$, we have $f_Z(z) = \frac{d}{dz}F_Z(z) = \frac{d}{dz}(1 - ((6 - z)^2/2)/9) = (6 - z)/9$.
Otherwise, we have $f_Z(z) = 0$.