STAT/MA 41600
In-Class Problem Set #32 part 2: October 30, 2017
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Problem Set 32 part 2 Answers

1a. We have $f_X(x) = \int_x^\infty \frac{3}{64} e^{-x/4-y/8}\,dy = (\frac{3}{64})(-8)e^{-x/4-y/8}|_{y=x}^\infty = \frac{3}{8}e^{-3x/8}$ for $x > 0$, and $f_X(x) = 0$ otherwise.
1b. We have $f_{Y|X}(y|5) = \frac{f_{X,Y}(5,y)}{f_X(5)} = \frac{\frac{3}{64}e^{-5/4-y/8}}{\frac{3}{8}e^{-(3)(5)/8}} = \frac{1}{8}e^{-(y-5)/8}$ for $y > 5$, and $f_{Y|X}(y|5) = 0$ otherwise.
1c. We have $P(Y > 12 \mid X = 5) = \int_{12}^\infty f_{Y|X}(y|5)\,dy = \int_{12}^\infty \frac{1}{8}e^{-(y-5)/8}\,dy = -e^{-(y-5)/8}|_{y=12}^\infty = e^{-7/8}$.

2. The probability is $1 - e^{-30/180} = 1 - e^{-1/6}$.

3. Let X and Y be the times that Linus and Sally fall asleep (given in minutes after midnight). The probability is $P(|X - Y| < 5) = P(Y < X < Y + 5) + P(X < Y < X + 5)$, and by symmetry, this equals $2P(X < Y < X + 5)$, i.e., we only need to calculate one of these probabilities, and then double our answer. So we get $2\int_0^\infty \int_x^{x+5} (1/400)e^{-x/20-y/20}\,dy\,dx = 2\int_0^\infty (-1/20)e^{-x/20-y/20}|_{y=x}^{x+5}\,dx = 2\int_0^\infty (-1/20)(e^{-x/20-(x+5)/20} - e^{-x/20-x/20})\,dx$ which simplifies to $2\int_0^\infty (1/20)(e^{-x/10} - e^{-x/10-1/4})\,dx = 2(-1/2)(e^{-x/10} - e^{-x/10-1/4})|_{x=0}^\infty = 1 - e^{-1/4}$

4. We compute $P(U < X) = \int_0^{10}\int_u^\infty (\frac{1}{10})(\frac{1}{5})e^{-x/5}\,dx\,du = \int_0^{10} (\frac{1}{10})(-e^{-x/5})|_{x=u}^\infty\,du = \int_0^{10} (\frac{1}{10})e^{-u/5}\,du = -\frac{1}{2}e^{-u/5}|_{u=0}^{10} = \frac{1}{2}(1 - e^{-2})$