Problem Set 39 Answers

1a. We have Cov\( (X_i, X_i) = \mathbb{E}(X_i X_i) - \mathbb{E}(X_i)\mathbb{E}(X_i) = \mathbb{E}(X_i) - \mathbb{E}(X_i)\mathbb{E}(X_i) = 4/2 - (4/52)^2 = 12/169. \)

1b. We have Cov\( (X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j) = (4/52)(3/51) - (4/52)^2 = -4/2873. \)

1c. We have Var\( (X) = Cov(X, X) = Cov(X_1 + \cdots + X_5, X_1 + \cdots + X_5) = 5 Cov(X_1, X_1) + 20 Cov(X_1, X_2) = 5(12/169) + 20(-4/2873) = 940/2873. \)

2a. We have Cov\( (X_i, X_i) = \mathbb{E}(X_i X_i) - \mathbb{E}(X_i)\mathbb{E}(X_i) = \mathbb{E}(X_i) - \mathbb{E}(X_i)\mathbb{E}(X_i) = 3/6 - (3/6)^2 = 1/4. \)

2b. We have Cov\( (X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j) = (3/6)(2/5) - (3/6)^2 = -1/20. \)

2c. We have Var\( (X) = Cov(X, X) = Cov(X_1 + \cdots + X_3, X_1 + \cdots + X_3) = 3 Cov(X_1, X_1) + 6 Cov(X_1, X_2) = 3(1/4) + 6(-1/20) = 9/20. \)

3. We have Cov\( (X, Y) = \int_0^2 \int_{y-4}^{8-y} (x - 4/3)(y - 2/3)(1/12) \, dx \, dy = \int_0^2 (x^2/2 - 4x/3)(y - 2/3)(1/12) \, dy = \int_0^2 (6y^2 - 16y + 8)(y - 2/3)(1/12) \, dy = \int_0^2 (6y^3 - 20y^2 + 56y/3 - 16/3)(1/12) \, dy = (3y^4/2 - 20y^3/3 + 28y^2/3 - 16y/3)|_{y=0}^{y=1/2} = -2/9. \)

4. We have \( \mathbb{E}(X) = \int_0^2 (3x^2/8) \, dx = 3x^4/32|_{x=0}^{x=2} = 3/2 \) and \( \mathbb{E}(Y) = \int_0^2 (y)(3/4)(2 - 2y + y^2/2) \, dy = (3/4)(y^2 - 2y^3/3 + y^4/8)|_{y=0}^{y=1/2} = 1/20. \)

So we conclude Cov\( (X, Y) = \int_0^2 \int_0^x (x - 3/2)(y - 1/2)(3/4)(x - y) \, dy \, dx = \int_0^2 (x - 3/2)(3/4) \int_0^x (xy - x/2 - y^2 + y/2) \, dy \, dx = \int_0^2 (x - 3/2)(3/4)(x^3 - 3x^2y/2 - xy^2/2 - y^3/3 + y^2/4)|_{y=0}^{y=x} \, dx = \int_0^2 (x - 3/2)(3/4)(x^3/6 - x^2/4) \, dx = \int_0^2 (3/4)(x^4/6 - x^3/2 + 3x^2/8) \, dx = (3/4)(x^5/30 - x^4/8 + x^3/8)|_{x=0}^{x=1/2} = 1/20. \)