STAT/MA 41600
In-Class Problem Set #40: November 27, 2017
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Problem Set 40 Answers

1. We first compute $f_Y(2) = \int_2^\infty 21e^{-3x-4(2)}\,dx = -7e^{-3x-8}|_{x=2}^\infty = 7e^{-14}$. This yields $f_{X|Y}(x \mid 2) = \frac{f_{X,Y}(x,2)}{f_Y(2)} = \frac{21e^{-3x-4(2)}}{7e^{-14}} = 3e^{-3x+6}$ for $x > 2$, and $f_{X|Y}(x \mid 2) = 0$ otherwise. Now we conclude that $\mathbb{E}(X \mid Y = 2) = \int_2^\infty (x)(3e^{-3x+6})\,dx$. There are (at least) a couple of ways to proceed, to evaluate this integral:

Method #1: We can use integration by parts, with $u = 3x$ and $dv = e^{-3x+6}\,dx$, so $du = 3\,dx$ and $v = e^{-3x+6}/(-3)$, which yields $\mathbb{E}(X \mid Y = 2) = (3x)(e^{-3x+6})/(-3)|_{x=2}^\infty - \int_2^\infty (e^{-3x+6}/(-3))(3)\,dx = 2 - \int_2^\infty e^{-3x+6}\,dx = 2 - e^{-3x+6}/(-3)|_{x=2}^\infty = 2 + 1/3 = 7/3$.

Method #2: We can use a u-substitution first, and then some facts about Exponential random variables. We use $u = x-2$ and $du = dx$ to get $\mathbb{E}(X \mid Y = 2) = \int_2^\infty (x)(3e^{-3(x-2)})dx = \int_0^\infty (u+2)(3e^{-3u})\,du = \int_0^\infty (u)(3e^{-3u})\,du + 2\int_0^\infty (3e^{-3u})\,du$. The first term is 1/3 because this is the expected value of an Exponential random variable with $\lambda = 3$. The second term is (2)(1) because we are integrating the probability density function of an Exponential random variable with $\lambda = 3$ over all values of u, and this integral is 1. So we conclude that $\mathbb{E}(X \mid Y = 2) = 1/3 + (2)(1) = 7/3$.

Please note that the second method only looks a little longer, because we explained the ideas with complete English sentences, but this second method is a little easier and does *not* require integration by parts.

2a. We use the concepts of Poisson splitting. The number of red bears, the number of blue bears, and the number of yellow bears are each Poisson random variables, each of which have mean 6, and *these three random variables are independent*. So the number of yellow bears is a Poisson random variable with $\lambda = 6$, regardless of knowing how many red and blue bears appear.

2b. The expected number of yellow bears is 6.

3. We have $p_Y(3) = p_{X,Y}(1,3) + p_{X,Y}(2,3) + p_{X,Y}(3,3) = \frac{1}{(7)(3)} + \frac{2}{(7)(3)} + \frac{3}{(7)(3)} = 6/21 = 2/7$.

So we obtain $p_{X|Y}(x \mid 3) = \frac{p_{X,Y}(x,3)}{p_Y(3)} = \frac{x/((7)(3))}{2/7} = x/6$ for $x = 1, 2, 3$, and $p_{X|Y}(x \mid 3) = 0$ otherwise. Then we conclude that $\mathbb{E}(X \mid Y = 3) = (1)(1/6) + (2)(2/6) + (3)(3/6) = 7/3$.

4a. We compute $f_Y(1) = \int_{-2}^4 1/12\,dx = 1/2$, so we get $f_{X|Y}(x \mid 1) = \frac{f_{X,Y}(x,1)}{f_Y(1)} = \frac{1/12}{1/2} = 1/6$ for $-2 < x < 4$, and $f_{X|Y}(x \mid 1) = 0$ otherwise. So we conclude that $\mathbb{E}(X \mid Y = 1) = \int_{-2}^4 (x)(1/6)\,dx = (x^2/2)(1/6)|_{x=-2}^4 = 1$.

4b. We compute $f_Y(1/3) = \int_{1/3}^2 (3/4)(x - 1/3)\,dx = (3/4)(x^2/2 - x/3)|_{x=1/3}^2 = 25/24$, so we get $f_{X|Y}(x \mid 1/3) = \frac{f_{X,Y}(x,1/3)}{f_Y(1/3)} = \frac{(3/4)(x-1/3)}{25/24} = (18/25)(x - 1/3)$ for $1/3 < x < 2$, and $f_{X|Y}(x \mid 1/3) = 0$ otherwise. So we conclude that $\mathbb{E}(X \mid Y = 1/3) = \int_{1/3}^2 (x)(18/25)(x - 1/3)\,dx = \int_{1/3}^2 (18/25)(x^2 - x/3)\,dx = (18/25)(x^3/3 - x^2/6)|_{x=1/3}^2 = 13/9$.