STAT/MA 41600
In-Class Problem Set #1: August 22, 2018
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Problem Set 1 Answers

1a. The daddy bear is always in the same place. The next five places (counterclockwise, for instance) can be filled, respectively, in 5, 4, 3, 2, 1 ways. So there are $5! = 120$ outcomes.
1b. There are 4 places where the daughters can sit. Within the 2 chosen seats, the daughters have $2! = 2$ ways to sit, and within the remaining 3 seats, the sons have $3! = 6$ ways to sit. So there are $(4)(2)(6) = 48$ outcomes in which the daughters sit next to each other.
1c. Same method as above, except that there are only 2 places where the daughters can sit, if they want to leave room for the sons to sit together too. So there are $(2)(2)(6) = 24$ outcomes in which the daughters sit next to each other and the sons sit next to each other.

2a. Each pick has 6 possibilities, so there are $6^3 = 216$ possible outcomes altogether.
2b. There are 2^{216} events. (To build an event, decide whether or not each outcome is in the event.)
2c. If we avoid purple, then each pick has 5 possibilities, so there are $5^3 = 125$ possible outcomes without purple. So there are $216 - 125 = 91$ outcomes with at least one purple.

3a. The outcomes in A_1 have purple or orange as the 1st selection, and any of the 6 colors is OK for the 2nd and 3rd selections, so there are $(2)(6)(6) = 72$ such outcomes.
3b. The outcomes in $A_1 \cap A_2 \cap A_3$ have purple or orange on each selection, so there are $2^3 = 8$ such outcomes.
3c. The outcomes in $A_1 \cup A_2 \cup A_3$ have at least one purple or orange selection. The outcomes in the complementary event, $(A_1 \cup A_2 \cup A_3)^c = A_1^c \cap A_2^c \cap A_3^c$ have no purple or orange selections; there are $4^3 = 64$ such outcomes. So there are $216 - 64 = 152$ outcomes in $A_1 \cup A_2 \cup A_3$, each of which has at least one purple or orange selection.

4a. The exponential function has Taylor series $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$, which is valid for all real values x. In particular, we have $e^2 = \sum_{j=0}^{\infty} \frac{2^j}{j!}$. Subtracting the first three terms on both sides, we have $e^2 - \frac{2^0}{0!} - \frac{2^1}{1!} - \frac{2^2}{2!} = \sum_{j=3}^{\infty} \frac{2^j}{j!}$, or equivalently, $e^2 - 5 = \sum_{j=3}^{\infty} \frac{2^j}{j!}$.
4b. We use the fact that $\sum_{j=1}^{\infty}(1/3)^{j-1}(2/3)$ is a geometric series, with a ratio of $1/3$ between consecutive terms. In general, for $-1 < a < 1$, we have $\sum_{j=1}^{\infty} a^{j-1} = \frac{1}{1-a}$, so in this case, we compute $\sum_{j=1}^{\infty}(1/3)^{j-1}(2/3) = (2/3)\sum_{j=1}^{\infty}(1/3)^{j-1} = \frac{2/3}{1-1/3} = 1$. Alternatively, we can compute the value of the series directly, using telescoping, as follows:

$$\sum_{j=1}^{\infty}(1/3)^{j-1}(2/3) = (2/3)(1 + 1/3 + 1/3^2 + 1/3^3 + \cdots)$$
$$= (2/3)(1 + 1/3 + 1/3^2 + 1/3^3 + \cdots)\frac{(1-1/3)}{(1-1/3)} = \frac{(2/3)}{(1-1/3)} = 1.$$

Subtracting the first three terms, we get $\sum_{j=4}^{\infty}(1/3)^{j-1}(2/3) = 1 - (2/3)(1 + 1/3 + 1/3^2) = 1/27$.

Extra brain stretch: The series $\sum_{j=1}^{\infty} 1/j$ does not converge. To see this, just notice (e.g., by drawing a picture) that $\sum_{j=1}^{N} 1/j \geq \int_1^{N+1} 1/x\,dx = \ln(N+1)$ for all positive integers N. Since we have $\lim_{N\to\infty} \ln(N+1) = +\infty$, it follows that $\sum_{j=1}^{\infty} 1/j = +\infty$ too, i.e., that $\sum_{j=1}^{N} 1/j$ diverges (to infinity) for large N. For the other brain stretch: Since we have $\ln(1+x) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} x^j$ for $-1 < x \leq 1$, then for $x = 1$ the series converges and we get $\ln(2) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}$.