STAT/MA 41600
In-Class Problem Set #8: September 7, 2018
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Problem Set 8 Answers

1. The probability mass function of X is $p_X(1) = 1/2$ and $p_X(0) = 1/2$. This can be seen by symmetry between the blue and white, or we can calculate directly: $p_X(1) = (1/5)(4/4) + (1/5)(3/4) + (1/5)(2/4) + (1/5)(1/4) + (1/5)(0/4) = 1/2$.

2. For $x = 1, 2, 3, 4$, we have $p_X(x) = (1/5)(1/4) + (1/5)(1/6) + (1/5)(1/8) + (1/5)(1/12) + (1/5)(1/20) = 27/200$.

For $x = 5, 6$, we have $p_X(x) = (1/5)(0) + (1/5)(1/6) + (1/5)(1/8) + (1/5)(1/12) + (1/5)(1/20) = 17/200$.

For $x = 7, 8$, we have $p_X(x) = (1/5)(0)+(1/5)(0)+(1/5)(1/8)+(1/5)(1/12)+(1/5)(1/20) = 31/600$.

For $x = 9, 10, 11, 12$, we have $p_X(x) = (1/5)(0) + (1/5)(0) + (1/5)(0) + (1/5)(1/12) + (1/5)(1/20) = 2/75$.

For $x = 13, \ldots, 20$, we have $p_X(x) = (1/5)(0)+(1/5)(0)+(1/5)(0)+(1/5)(0)+(1/5)(1/20) = 1/100$.

Otherwise, we have $p_X(x) = 0$.

3a. The probability mass function of X is $p_X(x) = (40/52)^{x-1}(12/52)$ for integers $x \geq 1$, and $p_X(x) = 0$ otherwise.

3b. We compute $\sum_{x=1}^{\infty}(40/52)^{x-1}(12/52) = (12/52)(1+40/52+(40/52)^2+(40/52)^3+\cdots) = (12/52)/(1-40/52) = 1$.

4. Regardless of who is in the leftmost seat, their partner is next to them with probability 1/5. Then whoever is in the third seat, their partner is next to them with probability 1/3, and this forces the last couple to be together too (think: xxyyzz). So we obtain $p_X(3) = (1/5)(1/3) = 1/15$.

There are only 3 places that two happy bear pairs can sit, if the other bear pair remains unhappy (think: xxAyyA, AxxAyy, AxxyyA). Each such configuration has probability 1/15 of occurring (same reasoning as above). So we obtain $p_X(2) = (3)(1/15) = 1/5$.

There are only 6 places that 1 happy bear pair can sit and 2 unhappy bear pairs can sit (think: xxABAB, AxxBAB, ABxxAB, ABxxBA, ABAxxB, ABABxx). Each such configuration has probability 1/15 of occurring (same reasoning as above). So we obtain $p_X(1) = (6)(1/15) = 2/5$.

There are only 5 places that 3 unhappy bear pairs can sit (think: ABCABC, ABCACB, ABCBAC, ABCBCA ABACBC). Each such configuration has probability 1/15 of occurring (same reasoning as above). So we obtain $p_X(0) = (5)(1/15) = 1/3$.

We can verify that these probabilities sum to 1, by checking $1/15 + 1/5 + 2/5 + 1/3 = 1$.