Problem Set 9 Answers

1. The probability that \( Y \) is greater than or equal to \( X \) is

\[
\sum_{x=1}^{\infty} \sum_{y=x}^{\infty} \left( \frac{3}{4} \right)^{x-1} \left( \frac{1}{4} \right)^{y-1} \frac{48}{52} \sum_{y=1}^{\infty} \left( \frac{48}{52} \right)^{y-1},
\]

and we see that the inner sum is \( \sum_{y=1}^{\infty} \left( \frac{48}{52} \right)^{y-1} = \left( \frac{48}{52} \right)^{x} / (1 - \frac{48}{52}) \).

So the desired probability becomes \( \sum_{x=1}^{\infty} \left( \frac{3}{4} \right)^{x-1} \left( \frac{1}{4} \right)^{x-1} \frac{48}{52} \sum_{y=1}^{\infty} \left( \frac{48}{52} \right)^{y-1} = \left( \frac{48}{52} \right)^{x} / (1 - \frac{48}{52}) \).

The probability is \( \frac{1}{2} \).

2. The probability is \( \frac{1}{4} \) for the ace, \( \frac{1}{4} \) for the blue, \( \frac{1}{4} \) for the white, which implies \( \frac{1}{4} \).

3. We look at the 24 possible outcomes, and we compute \( p_X(0) = \frac{4}{24}, p_X(1) = \frac{7}{24}, p_X(2) = \frac{6}{24}, p_X(3) = \frac{4}{24}, p_X(4) = \frac{2}{24}, \) and \( p_X(5) = \frac{1}{24} \).

4a. We compute \( P(X > 4 \mid Y = 1) = \frac{P(X > 4 \cap Y = 1)}{P(X > 1 \cap Y = 1)} = \frac{\sum_{x=4}^{\infty} \left( \frac{11}{16} \right) \left( \frac{1}{4} \right)^{x-1} \left( \frac{1}{3} \right)^{y-1}}{\sum_{x=1}^{\infty} \left( \frac{11}{16} \right) \left( \frac{1}{4} \right)^{x-1} \left( \frac{1}{3} \right)^{y-1} \sum_{x=1}^{\infty} \left( \frac{1}{4} \right)^{x-1}} = \frac{(1/4)^{(11/16)}(1/4)}{1/(1/3)} = (1/4)^{11/16} = 1/256.

4b. The random variables \( X \) and \( Y \) are dependent because we need to have \( Y \geq X \) in this setup.

4c. For positive integers \( y \geq 1 \), we compute \( p_Y(y) = \sum_{x=y}^{\infty} \left( \frac{11}{16} \right) \left( \frac{1}{4} \right)^{x-1} \left( \frac{1}{3} \right)^{y-1} = (11/16)(1/3)^{y-1} \sum_{x=y}^{\infty} \left( \frac{1}{4} \right)^{x-1} = (11/16)(1/3)^{y-1} (1/4)^{y-1}/(1-1/4) = (11/12)(1/12)^{y-1} \), and \( p_Y(y) = 0 \) otherwise.

Here is another method for question 1, which does not require a double sum:

On any given flip, Bob gets blue and Cynthia gets an Ace with probability \( \frac{1}{4} \) (this implies \( X = Y \)); or Bob gets blue and Cynthia does not get an Ace with probability \( \frac{1}{4} \) (this implies \( Y \geq X \)); or Bob gets white and Cynthia gets an Ace with probability \( \frac{3}{4} \) (this implies \( X > Y \)). If Bob gets white and Cynthia does not get an Ace, they just proceed to another round. So we conclude that \( P(Y \geq X) = (1/4)(4/52) + (1/4)(48/52) = 13/16. \)