Problem Set 12 Answers


1b. The twenty five terms are $X^2 = (X_1 + \cdots + X_5)^2 = X_1X_1 + X_1X_2 + \cdots + X_5X_5$, so we get $E(X^2) = E((X_1 + \cdots + X_5)^2) = E(X_1X_1) + E(X_1X_2) + \cdots + E(X_5X_5)$. By symmetry, this simplifies to $E(X^2) = 5E(X_1X_1) + 20E(X_1X_2)$. We have $X_1X_1 = X_1$ (since $X_1$ is an indicator random variable), so we get $E(X_1X_1) = E(X_1) = 12/52$. We also have $E(X_1X_2) = (12/52)^2$. So altogether we get $E(X^2) = (5)(12/52) + (20)(12/52)^2 = 375/169$.

1c. We get $Var(X) = E(X^2) - (EX)^2 = 375/169 - (15/13)^2 = 150/169$.

1d. Since the $X_j$’s are independent, we can add the variances, i.e., $Var(X) = Var(X_1 + \cdots + X_5) = Var(X_1) + \cdots + Var(X_5)$, which equals $5Var(X_1)$ by symmetry. We have $Var(X_1) = E(X_1^2) - (E(X_1))^2 = E(X_1) - (E(X_1))^2 = 12/52 - (12/52)^2 = 30/169$. So we conclude that $Var(X) = (5)(30/169) = 150/169$.

2a. We have $E(X^2) = (0^2)(2109/8330) + (1^2)(703/1666) + (2^2)(209/833) + (3^2)(55/833) + (4^2)(165/21658) + (5^2)(33/108290) = 475/221.

2b. The twenty five terms are $X^2 = (X_1 + \cdots + X_5)^2 = X_1X_1 + X_1X_2 + \cdots + X_5X_5$, so we get $E(X^2) = E((X_1 + \cdots + X_5)^2) = E(X_1X_1) + E(X_1X_2) + \cdots + E(X_5X_5)$. By symmetry, this simplifies to $E(X^2) = 5E(X_1X_1) + 20E(X_1X_2)$. We have $X_1X_1 = X_1$ (since $X_1$ is an indicator random variable), so we get $E(X_1X_1) = E(X_1) = 12/52$. We also have $E(X_1X_2) = (12/52)(11/51)$. So altogether we get $E(X^2) = (5)(12/52) + (20)(12/52)(11/51) = 475/221$.

2c. We get $Var(X) = E(X^2) - (EX)^2 = 475/221 - (15/13)^2 = 2350/2873$.

3a. We have $E(X^2) = (1^2)(37/64) + (2^2)(19/64) + (3^2)(7/64) + (4^2)(1/64) = 3$.

3b. We have $E(X^2) = E((X_1 + \cdots + X_4)^2)$, which has:
6 terms of the form $E(X_iX_j) = E(X_j)$ for $i < 4$; 4 terms of the form $E(X_iX_5) = E(X_5)$ for $i < 3$;
2 terms of the form $E(X_iX_2) = E(X_2)$ for $i < 2$; and of course the terms of the form $E(X_2^2) = E(X_2)$. So we get $E(X^2) = (7)(1/4)^3 + (5)(2/4)^3 + (3)(3/4)^3 + (1)(1) = 3$.

3c. We get $Var(X) = E(X^2) - (EX)^2 = 3 - (25/16)^2 = 143/256$.

4a. We have $E(X^2) = (0^2)(1/3) + (1^2)(2/5) + (2^2)(1/5) + (3^2)(1/15) = 9/5$.

4b. Methods #1 and #2 have nine terms that are $X^2 = (X_1 + X_2 + X_3)^2 = X_1X_1 + X_1X_2 + \cdots + X_3X_3$, so we get $E(X^2) = E((X_1 + X_2 + X_3)^2) = E(X_1X_1) + E(X_1X_2) + \cdots + E(X_3X_3)$. By symmetry, this simplifies to $E(X^2) = 3E(X_1X_1) + 6E(X_1X_2)$. We have $X_1X_1 = X_1$ (since $X_1$ is an indicator random variable), so we get $E(X_1X_1) = E(X_1) = 1/3$. To compute $E(X_1X_2)$, note that there are 12 places where these two bear pairs can sit, each of which has probability $(2/6)(1/5)(2/4)(1/3) = 1/90$, and thus $E(X_1X_2) = 12/90$. So altogether we get $E(X^2) = (3)(1/3) + (6)(12/90) = 9/5$.

Method #3 has twenty five terms that are $X^2 = (X_1 + \cdots + X_5)^2 = X_1X_1 + X_1X_2 + \cdots + X_5X_5$, so we get $E(X^2) = E((X_1 + \cdots + X_5)^2) = E(X_1X_1) + E(X_1X_2) + \cdots + E(X_5X_5)$. There are 5 elements of the form $E(X_1X_1)$ (all of which are $E(X_1)$). There are 8 elements of the form $E(X_jX_{j+1})$ (all of which are zero in this case). There are 12 elements of the form $E(X_iX_j)$ in which $i$ and $j$ are (strictly) more than 1 apart (all of which are, by symmetry, the same). So this simplifies to $E(X^2) = 5E(X_1) + 8(0) + 12E(X_1X_3)$. We know $E(X_1) = 1/5$. We have $E(X_1X_3) = (1/5)(1/3) = 1/15$. So altogether we get $E(X^2) = (5)(1/5) + (8)(0) + (12)(1/15) = 9/5$.

4c. We get $Var(X) = E(X^2) - (EX)^2 = 9/5 - (1)^2 = 4/5$. 