We compute \( E(X_1 + \cdots + X_8) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_8) = 8\mathbb{E}(X_1) = (8)(1/8) = 1. \)

1b. We compute \( \mathbb{E}((X_1 + \cdots + X_8)^2) = 8\mathbb{E}(X_1^2) + 56\mathbb{E}(X_1X_2). \) As before, we have \( \mathbb{E}(X_1X_2) = \mathbb{E}(X_1) = 1/8. \) We also notice that \( \mathbb{E}(X_1X_2) = (1/8)(1/7). \) So we conclude that \( \text{Var}(X_1 + \cdots + X_8) = ((8)(1/8) + (56)(1/8)(1/7)) - (1)^2 = 1 + 1 - 1 = 1. \)

2a. We compute \( P(Y = X) = \sum_{x=1}^{5} (1/5)(2/3)^{x-1}(1/3) = (1/5)(1/3)(1 - (2/3)^5)/(1 - 2/3) = (1/5)(1 - (2/3)^5) = 211/1215 = 0.1737. \)

2b. We compute \( P(Y > X) = \sum_{x=1}^{5} (1/5)(2/3)^x = (1/5)(2/3 - (2/3)^6)/(1 - 2/3) = 422/1215 = 0.3473. \)

3a. We compute that \( P(Y = X) = \sum_{x=1}^{\infty} e^{-5x/2}(2/3)^{x-1}(1/3) = (e^{-5})(1/3) \sum_{x=1}^{\infty} \frac{5x}{2} e^{5x/2}(2/3)^x = (e^{-5})(1/2) \sum_{x=1}^{\infty} \frac{(10/3)^x}{x!} = (e^{-5})(1/2)(e^{10/3} - 1) = 0.0911. \)

3b. We get \( P(Y > X) = \sum_{x=0}^{\infty} \sum_{y=1}^{x} e^{-5x/2}(2/3)^y(1/3) = \sum_{x=0}^{\infty} \sum_{y=1}^{\infty} e^{-5y/2}(2/3)^y = \sum_{x=0}^{\infty} e^{-5y/2}(2/3)^y = \sum_{x=0}^{\infty} e^{-5y/2}(10/3)^y = (e^{-5})(e^{10/3}) = e^{-5/3} = 0.1889. \)

4. In all cases below, we let \( X_j = 1 \) if the \( j \)th person is happy, and \( X_j = 0 \) otherwise. We let \( X \) denote the total number of happy students. So we have \( X = X_1 + \cdots + X_7. \)

4a. We have \( \mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_7) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_7) = 1/7 + \cdots + 1/7 = 1. \)

4b. We have \( \text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2. \) We compute \( \mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_7)^2) = 7\mathbb{E}(X_1^2) + 42\mathbb{E}(X_1X_2) = 7\mathbb{E}(X_1) + 42\mathbb{E}(X_1X_2) = 7(1/7) + 42(0) = 1. \) So we conclude that \( \text{Var}(X) = 1 - 1^2 = 0. \)

Note: A different way to derive the answer to 4a and 4b is to note that exactly 1 student will always get the blue pencil, so there is always exactly 1 happy student, so the expected number of happy students is 1 and the variance of the number of happy students is 0 (because the random variable is just a constant 1 (always) in that setup).

4c. We have \( \mathbb{E}(X) = 1 \) for the exact same reasoning as in 4a.

4d. We have \( \text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2. \) We compute \( \mathbb{E}(X^2) = \mathbb{E}((X_1 + \cdots + X_7)^2) = 7\mathbb{E}(X_1^2) + 42\mathbb{E}(X_1X_2) = 7\mathbb{E}(X_1) + 42\mathbb{E}(X_1X_2) = 7(1/7) + 42(1/7)(1/6) = 2. \) So we conclude that \( \text{Var}(X) = 2 - 1^2 = 1. \)