STAT/MA 41600
In-Class Problem Set #29: October 22, 2018
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Problem Set 29 Answers

1. We have $\mathbb{E}(X^2) = \int_0^\infty \int_x^\infty (x^2)(\frac{1}{750}e^{-(x/150+y/30)})\,dy\,dx = \int_0^\infty (x^2)(-\frac{1}{25}e^{-(x/150+y/30)})|_{y=x}^\infty\,dx = \int_0^\infty (x^2)(\frac{1}{25}e^{-x/25})\,dx$. Then we use integration by parts, with $u = x^2$ and $dv = \frac{1}{25}e^{-x/25}\,dx$. We get $\mathbb{E}(X^2) = (x^2)(-e^{-x/25})|_{x=0}^\infty - \int_0^\infty (2x)(-e^{-x/25})\,dx = \int_0^\infty (2x)(e^{-x/25})\,dx$. Then we can pull out 2, and we can multiply and divide by 25, so we get $\mathbb{E}(X^2) = (2)(25)\int_0^\infty (x)(1/25)(e^{-x/25})\,dx$. We could do integration by parts again, or we could recognize that we already computed this integral in the previous problem set, and its value is 25, so we have $\mathbb{E}(X^2) = (2)(25)(25) = 1250$. Then we conclude that $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 1250 - 25^2 = 25^2 = 625$.

Alternatively, we could use the probability density function of X, which we derived in Problem Set 25, question 2. This saves us from having to compute line 1 of the paragraph above. We (instead) directly get the fact that $\mathbb{E}(X^2) = \int_0^\infty (x^2)(\frac{1}{25}e^{-x/25})\,dx$, and we use integration by parts (starting exactly at the second line of the first paragraph, above) to conclude that $\mathbb{E}(X^2) = 25$ and $\text{Var}(X) = 625$.

2. We compute as follows $\mathbb{E}(X^2) = \int_0^{10}\int_0^{-3x/5+6}(x^2)(1/30)\,dy\,dx = \int_0^{10}(x^2)(1/30)(y)|_{y=0}^{-3x/5+6}\,dx = \int_0^{10}(x^2)(1/30)(-3x/5+6)\,dx = \int_0^{10}(1/30)(-3x^3/5+6x^2)\,dx = (1/30)(-3x^4/20+2x^3)|_{x=0}^{10} = 50/3$.

Then we conclude that $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 50/3 - (10/3)^2 = 50/9$.

Alternative, we could find the probability density function of X, $f_X(x) = \int_0^{-3x/5+6} 1/30\,dy = (1/30)(-3x/5+6)$ for $x > 0$, and $f_X(x) = 0$ otherwise. So we get $\mathbb{E}(X^2) = \int_0^{10}(x^2)(1/30)(-3x/5+6)\,dx$ and we finish the computation in the same way as above.

3. We know $k = 1/36$, from P.S. 26, question 4, so $\mathbb{E}(Y^2) = \int_0^4\int_0^3 (y^2)(1/36)(3-x)(4-y)\,dx\,dy = \int_0^4 (y^2)(1/36)(3x-x^2/2)|_{x=0}^3(4-y)\,dy = \int_0^4 (y^2)(1/8)(4-y)\,dy = \int_0^4 (1/8)(4y^2-y^3)\,dy = (1/8)(4y^3/3-y^4/4)|_{y=0}^4 = 8/3$. We conclude that $\text{Var}(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 8/3 - (4/3)^2 = 8/9$.

An alternative method is to use the results of Problem Set, question 4d, namely, that $f_Y(y) = (1/8)(4-y)$ for $0 < y < 4$, and $f_Y(y) = 0$ otherwise. Using this probability density function of Y, we get $\mathbb{E}(Y^2) = \int_0^4 (y^2)(1/8)(4-y)\,dy$ and we finish the computation in the same way as above.

4a. We compute as follows $\mathbb{E}(X^2) = \int_0^2\int_0^2 (x^2)(9/64)x^2y^2\,dy\,dx = \int_0^2 (x^2)(9/64)(x^2)(y^3/3)|_{y=0}^2\,dx = \int_0^2 (x^2)(9/64)(x^2)(8/3)\,dx = \int_0^2 (9/64)(x^4)(8/3)\,dx = 12/5$, so $\text{Var}(X) = 12/5 - (3/2)^2 = 3/20$.
4b. Using the probability density function, we get $\mathbb{E}(X^2) = \int_0^2 (x^2)(3/8)(x^2)\,dx = \int_0^2 (3/8)(x^4)\,dx = (3/8)(x^5/5)|_{x=0}^2 = 12/5$, so again $\text{Var}(X) = 12/5 - (3/2)^2 = 3/20$.