3. As before, we compute Cov($X_1, X_2$) = $E(X_1X_2) - E(X_1)E(X_2) = 12/90 - (1/3)^2 = 1/45$. We also have Var($X_1$) = Var($X_2$) = $E(X_1^2) - (E(X_1))^2 = 13/21 - (3/13)^2 = 1/3 - (1/3)^2 = 2/9$. So altogether we have $\rho(X_1, X_2) = Cov(X_1, X_2)/\sqrt{Var(X_1)Var(X_2)} = (1/45)/\sqrt{(2/9)(2/9)} = 1/10$. 

In-Class Problem Set #39 part 2: November 16, 2018

Solutions by Mark Daniel Ward

Problem Set 39 part 2 Answers

1. We have $E(XY) = \int_0^\infty \int_x^\infty (xy)(1/750)e^{-(x/150+y/30)} dy dx$. For the inner integral (only), integration by parts with $u = y$ and $dv = (x)(1/750)e^{-(x/150+y/30)}$, and thus $du = dy$ and $v = -(x)(1/25)e^{-(x/150+y/30)}$ gives us $\int_x^\infty (xy)(1/750)e^{-(x/150+y/30)} dy = -(xy)(1/25)e^{-(x/150+y/30)}|_y=x \int_x^\infty -(x)(1/25)e^{-(x/150+y/30)} dy = (x^2)(1/25)e^{-x/25} + (x)(6/5)e^{-x/25}.$

So altogether we have $E(XY) = \int_0^\infty ((x^2)(1/25)e^{-x/25} + (x)(6/5)e^{-x/25}) dx$. The first term is equal to $(2)(25^2)$, since it is equal to $E(X^2)$, where $X$ is an exponential random variable with $\lambda = 1/25$. The second term can be rewritten as $\int_0^\infty (x)(6/5)e^{-x/25} dx = 30 \int_0^\infty (x)(1/25)e^{-x/25} dx$, which equals $30(25)$, by again using our knowledge of the mean of an exponential random variable with $\lambda = 1/25$. So altogether we have $E(XY) = (2)(25^2) + (30)(25) = 2000$.

2a. For the inner integral (only), we get

\[
\int_x^\infty (x)(1/750)e^{-(x/150+y/30)} dy = -(x)(1/25)e^{-(x/150+y/30)}|_y=x \int_x^\infty (x)(1/25)e^{-(x/150+y/30)} dy = (x)(1/25)e^{-x/25}.
\]

Now we have $E(X) = \int_0^\infty (x^2)(1/25)e^{-x/25} dx = 25$, using our knowledge about the mean of exponential random variables.

2b. We have $E(Y) = \int_0^\infty \int_x^\infty (y)(1/750)e^{-(x/150+y/30)} dy dx$.

For the inner integral (only), integration by parts with $u = y$ and $dv = (1/750)e^{-(x/150+y/30)}$, and thus $du = dy$ and $v = -(1/25)e^{-(x/150+y/30)}$ gives us $\int_x^\infty (y)(1/750)e^{-(x/150+y/30)} dy = -(y)(1/25)e^{-(x/150+y/30)}|_y=x \int_x^\infty -(1/25)e^{-(x/150+y/30)} dy = (y)(1/25)e^{-y/25} + (6/5)e^{-x/25}.$

So we get $E(Y) = \int_0^\infty (((y)(1/25)e^{-y/25} + (6/5)e^{-x/25}) dx$. The first term is 25 (by using our knowledge about the mean of exponential random variables). The second term is $\int_0^\infty (6/5)e^{-x/25} dx = 30 \int_0^\infty (1/25)e^{-x/25} dx = 30$. So altogether we have $E(Y) = 25 + 30 = 55$.

3. As before, we compute $Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2) = (12/52)(11/51) - (3/13)^2 = -10/2873$. We also have $Var(X_1) = Var(X_2) = E(X_1^2) - (E(X_1))^2 = E(X_1) - (E(X_1))^2 = 3/13 - (3/13)^2 = 30/169$. So altogether we have $\rho(X_1, X_2) = Cov(X_1, X_2)/\sqrt{Var(X_1)Var(X_2)} = (-10/2873)/\sqrt{(30/169)(30/169)} = -1/51$.

4. As before, we compute $Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2) = 12/90 - (1/3)^2 = 1/45$. We also have $Var(X_1) = Var(X_2) = E(X_1^2) - (E(X_1))^2 = E(X_1) - (E(X_1))^2 = 1/3 - (1/3)^2 = 2/9$. So altogether we have $\rho(X_1, X_2) = Cov(X_1, X_2)/\sqrt{Var(X_1)Var(X_2)} = (1/45)/\sqrt{(2/9)(2/9)} = 1/10$. 