1. The probability mass function \( p_Y(y) \) of \( Y \) evaluated at \( y = 1 \) is 

\[
p_Y(1) = \sum_{x=1}^{\infty} (11/16)(1/4)^{x-1}(1/3)^{1-x} = (11/16)/(1-1/4) = 11/12.
\]

So the conditional probability mass function of \( X \), given \( Y = 1 \), is 

\[
p_{X|Y}(x \mid 1) = \frac{p_{X,Y}(x,1)}{p_Y(1)} = \frac{(11/16)(1/4)^{x-1}(1/3)^{1-x}}{11/12} = (3/4)(1/4)^{x-1} \text{ for } x \geq 1, \text{ and } p_{X|Y}(x \mid 1) = 0 \text{ otherwise.}
\]

Therefore, given \( Y = 1 \), the conditional distribution of \( X \) is Geometric with \( p = 3/4 \), so \( \mathbb{E}(X \mid Y = 1) = 1/p = 4/3 \).

2. The probability density function \( f_Y(y) \) of \( Y \) evaluated at \( y = 1 \) is 

\[
f_Y(1) = \int_0^{25/3} 1/30 \, dx.
\]

(The upper limit on the integral comes from the fact that the hypotenuse of the triangle is the line \( y = -(6/10)x + 6 \), i.e., \( x = 10 - (10/6)y \), so when \( y = 1 \) on this hypotenuse, we have \( x = 10 - (10/6) = 50/6 = 25/3 \).)

Therefore, we get \( f_Y(1) = (25/3)(1/30) = 25/90 = 5/18 \). So the conditional probability density function of \( X \), given \( Y = 1 \), is 

\[
f_{X|Y}(x \mid 1) = \frac{f_{X,Y}(x,1)}{f_Y(1)} = \frac{1/30}{5/18} = 3/25 \text{ for } 0 \leq x \leq 25/3, \text{ and } f_{X|Y}(x \mid 1) = 0 \text{ otherwise.}
\]

Therefore, given \( Y = 1 \), the conditional distribution of \( X \) is Continuous Uniform on \([0, 25/3]\), so \( \mathbb{E}(X \mid Y = 1) = (0 + 25/3)/2 = 25/6 \).

3. The probability density function \( f_X(x) \) of \( X \) evaluated at \( x = 20 \) is:

\[
f_X(20) = \int_0^{\infty} (1/750)e^{-(20/150+y/30)}dy = -(1/25)e^{-(2/15+y/30)}|_{y=20} = (1/25)e^{-(2/15+20/30)} = (1/25)e^{-4/5}.
\]

So the conditional probability density function of \( Y \), given \( X = 20 \), is 

\[
f_{Y|X}(y \mid 20) = \frac{f_{X,Y}(20,y)}{f_X(20)} = \frac{(1/750)e^{-(20/150+y/30)}}{(1/25)e^{-4/5}} = (1/30)e^{-(y-20)/30} \text{ for } y > 20, \text{ and } f_{Y|X}(y \mid 20) = 0 \text{ otherwise.}
\]

Therefore, given \( X = 20 \), the conditional expected value of \( Y \) is 

\[
\mathbb{E}(Y \mid X = 20) = \int_0^{\infty} (y)(1/30)e^{-(y-20)/30}dy = \int_0^{\infty} (y+20)(1/30)e^{-y/30}dy.
\]

We know that \( \int_0^{\infty} (y)(1/30)e^{-y/30}dy = 30 \) (this is the expected value of an Exponential random variable with \( \lambda = 1/30 \)), and we know that \( \int_0^{\infty} (20)(1/30)e^{-y/30}dy = 20 \int_0^{\infty} (1/30)e^{-y/30}dy = 20 \) (this is just integrating the probability density function of an Exponential random variable with \( \lambda = 1/30 \), and then multiplying by 20). So we conclude that \( \mathbb{E}(Y \mid X = 20) = 30 + 20 = 50 \).

Alternatively, we could have used integration by parts.

4. Let \( X \) denote the value on the blue die, and let \( Y \) denote the value of the sum of the two dice.

The probability mass function \( p_Y(y) \) of \( Y \) evaluated at \( y = 9 \) is \( p_Y(9) = 4/36 \), since exactly 4 of the 36 equally likely results for the pair of dice will make the sum \( Y \) be equal to 9.

So the conditional probability mass function of \( X \), given \( Y = 9 \), is 

\[
p_{X|Y}(x \mid 9) = \frac{p_{X,Y}(x,9)}{p_Y(9)} = \frac{1/36}{4/36} = 1/4 \text{ for } x = 3, 4, 5, 6, \text{ and } p_{X|Y}(x \mid 9) = 0 \text{ otherwise.}
\]

Therefore, given \( Y = 9 \), we compute 

\[
\mathbb{E}(X \mid Y = 9) = (1/4)(3) + (1/4)(4) + (1/4)(5) + (1/4)(6) = 9/2.
\]