STAT/MA 41600
In-Class Problem Set #42: November 30, 2018
Solutions by Mark Daniel Ward

Problem Set 42 Answers

1. The minimum of the five independent exponential random variables is also an exponential random variable, with parameter $\lambda = 1/2 + \cdots + 1/2 = 5/2$. So $P(X_{(1)} < 1.5) = F_{X_{(1)}}(1.5) = 1 - e^{-(5/2)(1.5)} = 1 - e^{-15/4} = 0.9765$.

2a. We note that $P(X_{(5)} \leq 20) = P(X_1, \ldots, X_5 \leq 20) = P(X_1 \leq 20) \cdots P(X_5 \leq 20) = (20/30) \cdots (20/30) = (2/3)^5 = 0.1317$.

2b. We note that $P(X_{(5)} \geq 25) = 1 - P(X_{(5)} \leq 25) = 1 - P(X_1, \ldots, X_5 \leq 25) = 1 - P(X_1 \leq 25) \cdots P(X_5 \leq 25) = 1 - (25/30) \cdots (25/30) = 1 - (5/6)^5 = 0.5981$.

2c. The density is $f_{X_{(3)}}(x) = \binom{5}{2,1,2}(1/30)(x/30)^2(1 - x/30)^2$ for $0 \leq x \leq 30$, and $f_{X_{(3)}}(x) = 0$ otherwise.

2d. We compute $\mathbb{E}(X_{(3)}) = \int_0^{30}(x)\binom{5}{2,1,2}(1/30)(x/30)^2(1 - x/30)^2\,dx = \int_0^{30}(x)(x/30)^2(1 - x/15 + x^2/900)\,dx = \int_0^{30}(x^3/900 - x^4/13500 + x^5/810000)\,dx = (x^4/3600 - x^5/67500 + x^6/4860000)|_{x=0}^{30} = 15$.

3. We have $f_{X_{(3)}}(x) = \binom{3}{2,1,0}(x^2/9)(x^3/27)^2(1 - x^3/27)^0 = x^8/2187$ for $0 < x < 3$, so we get $\mathbb{E}(X_{(3)}) = \int_0^3(x)(x^8/2187)\,dx = \int_0^3(x^9/2187)\,dx = (x^{10}/21870)|_{x=0}^3 = (3^{10}/21870) = 27/10$.

4. Let $X_j = 1$ if the jth card is a diamond, or $X_j = 0$ otherwise. Then we get:
$\mathrm{Cov}(X_1 + \cdots + X_5, X_1 + \cdots + X_5) = 5\mathrm{Cov}(X_1, X_1) + 20\mathrm{Cov}(X_1, X_2) = 5(\mathbb{E}(X_1^2) - (\mathbb{E}(X_1))^2) + 20(\mathbb{E}(X_1X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2)) = 5(13/52 - (13/52)^2) + 20((13/52)(12/51) - (13/52)^2) = 235/272 = 0.8640$.