STAT/MA 41600
In-Class Problem Set #44: December 5, 2018
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Problem Set 44 Answers

1a. We have $0 < X < 1$, so $\ln(X) < 0$, so $Y = -(1/7)\ln(X) > 0$.
1b. For $a \le 0$, the CDF of Y is $F_Y(a) = 0$. For $a > 0$, the CDF of Y is $F_Y(a) = P(Y \le a) = P(-(1/7)\ln(X) \le a) = P(\ln(X) \ge -7a) = P(X \ge e^{-7a} = 1 - P(X \le e^{-7a}) = 1 - e^{-7a}$, where the last equality holds since X is uniform on $(0, 1)$.
1c. Since the CDF of Y is $F_Y(a) = 1 - e^{-7a}$ for $a > 0$, and $F_Y(a) = 0$ otherwise, we recognize that Y is an Exponential random variable with parameter $\lambda = 7$.

2a. We have $X > 0$, so $-5X < 0$, and $Y = e^{-5X}$, so $0 < Y < 1$.
2b. For $a \le 0$, the CDF of Y is $F_Y(a) = 0$. For $a \ge 1$, the CDF of Y is $F_Y(a) = 1$. For $0 < a < 1$, the CDF of Y is $F_Y(a) = P(Y \le a) = P(e^{-5X} \le a) = P(-5X \le \ln a) = P(X \ge \ln a/(-5)) = 1 - P(X \le \ln a/(-5))$ but X is an Exponential random variable with $\lambda = 5$, so this is equal to $1 - (1 - e^{(-5)(\ln a/(-5))}) = e^{\ln a} = a$.
2c. Since the CDF of Y is $F_Y(a) = a$ for $0 < a < 1$, and $F_Y(a) = 0$ for $a \le 0$, and $F_Y(a) = 1$ for $a \ge 1$, we recognize that Y is a continuous uniform random variable on the interval $(0, 1)$.

3a. Yes, in this case, Y will be an Exponential random variable too. This is just a rescaling of X. For $a > 0$, the CDF of Y is $F_Y(a) = P(Y \le a) = P(cX \le a) = P(X \le a/c) = 1 - e^{-(\lambda)(a/c)} = 1 - e^{-(\lambda/c)(a)}$, so Y is an Exponential random variable with parameter λ/c, i.e., with $\mathbb{E}(Y) = c/\lambda$.

3b. Yes, Y will be a Normal random variable too. We are again just rescaling the random variable X. We have $P(Y \le a) = P(cX \le a) = P(X \le a/c) = \int_{-\infty}^{a/c} \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp(-(x-\mu_X)^2/(2\sigma_X^2))\,dx$. Then we use a substitution with $y = cx$ and $dy = c\,dx$. So we get

$$P(Y \le a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp(-(y/c - \mu_X)^2/(2\sigma_X^2))\,(1/c)\,dy,$$

which we can rewrite as

$$P(Y \le a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi(c\sigma_X)^2}} \exp(-(y - c\mu_X)^2/(2(c\sigma_X)^2))\,dy.$$

Therefore, Y is a normal random variable with mean $c\mu_X$ and variance $(c\sigma_X)^2 = c^2\sigma_X^2$.

3c. Yes, Y is a continuous Uniform random variable on $(0, cb)$. For $0 < a < cb$, we have $P(Y \le a) = P(cX \le a) = P(X \le a/c) = \frac{a/c}{b} = \frac{a}{cb}$, so indeed Y is uniformly distributed on $(0, cb)$.

4. We know that there are either 0 or 1 happy bears of each color. So let $X_j = 1$ if the jth color has a happy bear, and $X_j = 0$ otherwise. Then $\mathbb{E}(X_1 + \cdots + X_6) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_6) = 6\mathbb{E}(X_1) = (6)(3)(2/17)(1/16) = 9/68 = 0.1324$.

We also calculate $\mathrm{Var}(X_1 + \cdots + X_6) = \mathrm{Cov}(X_1 + \cdots + X_6, X_1 + \cdots + X_6) = 6\mathrm{Cov}(X_1, X_1) + 30\mathrm{Cov}(X_1, X_2) = 6(\mathbb{E}(X_1^2) - \mathbb{E}(X_1)\mathbb{E}(X_1)) + 30(\mathbb{E}(X_1X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2)) = 6((3)(2/17)(1/16) - (3)(2/17)(1/16)(3)(2/17)(1/16)) + 30((3)(2/17)(1/16)(13)/\binom{15}{3} - (3)(2/17)(1/16)(3)(2/17)(1/16)) = 4329/32368 = 0.1337$.