In-Class Problem Set #44: December 5, 2018

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Problem Set 44 Answers

1a. We have $0 < X < 1$, so $\ln(X) < 0$, so $Y = -(1/7) \ln(X) > 0$.

1b. For $a \leq 0$, the CDF of $Y$ is $F_Y(a) = 0$. For $a > 0$, the CDF of $Y$ is $F_Y(a) = P(Y \leq a) = P((-1/7)\ln(X) \leq a) = P(\ln(X) \geq -7a) = P(X \geq e^{-7a} = 1 - P(X \leq e^{-7a}) = 1 - e^{-7a}$, where the last equality holds since $X$ is uniform on $(0,1)$.

1c. Since the CDF of $Y$ is $F_Y(a) = 1 - e^{-7a}$ for $a > 0$, and $F_Y(a) = 0$ otherwise, we recognize that $Y$ is an Exponential random variable with parameter $\lambda = 7$.

2a. We have $X > 0$, so $-5X < 0$, and $Y = e^{-5X}$, so $0 < Y < 1$.

2b. For $a \leq 0$, the CDF of $Y$ is $F_Y(a) = 0$. For $a \geq 1$, the CDF of $Y$ is $F_Y(a) = 1$. For $0 < a < 1$, the CDF of $Y$ is $F_Y(a) = P(0 < Y < a) = P(e^{-5X} < a) = P(-5X < \ln a) = P(X \geq \ln a /(-5)) = 1 - P(X \leq \ln a /(-5))$ but $X$ is an Exponential random variable with $\lambda = 5$, so this is equal to $1 - (1 - e^{(-5)(\ln a /(-5))}) = e^{\ln a} = a$.

2c. Since the CDF of $Y$ is $F_Y(a) = a$ for $0 < a < 1$, and $F_Y(a) = 0$ for $a \geq 0$, and $F_Y(a) = 1$ for $a \geq 1$, we recognize that $Y$ is a continuous uniform random variable on the interval $(0,1)$.

3a. Yes, in this case, $Y$ will be an Exponential random variable too. This is just a rescaling of $X$. For $a > 0$, the CDF of $Y$ is $F_Y(a) = P(0 < Y < a) = P(cX < a) = P(X \leq a/c) = 1 - e^{-(\lambda /c)(a/c)} = 1 - e^{-(\lambda /c)(a)}$, so $Y$ is an Exponential random variable with parameter $\lambda /c$, i.e., with $E(Y) = c /\lambda$.

3b. Yes, $Y$ will be a Normal random variable too. We are again just rescaling the random variable $X$. We have $P(Y \leq a) = P(cX \leq a) = P(X \leq a/c) = \int_{-\infty}^{a/c} \frac{1}{\sqrt{2\pi \sigma_X^2}} \exp(-(x - \mu_X)^2/(2\sigma_X^2)) \, dx$. Then we use a substitution with $y = cx$ and $dy = c \, dx$. So we get

$$P(Y \leq a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi \sigma_X^2}} \exp(-(y/c - \mu_X)^2/(2\sigma_X^2)) \, (1/c) \, dy,$$

which we can rewrite as

$$P(Y \leq a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi (c\sigma_X)^2}} \exp(-(y - c\mu_X)^2/(2(c\sigma_X)^2)) \, dy.$$

Therefore, $Y$ is a normal random variable with mean $c \mu_X$ and variance $(c\sigma_X)^2 = c^2\sigma_X^2$.

3c. Yes, $Y$ is a continuous Uniform random variable on $(0, cb)$. For $0 < a < cb$, we have $P(Y \leq a) = P(cX \leq a) = P(X \leq a/c) = a/c = a/cb$, so indeed $Y$ is uniformly distributed on $(0, cb)$.

4. We know that there are either 0 or 1 happy bears of each color. So let $X_j = 1$ if the $j$th color has a happy bear, and $X_j = 0$ otherwise. Then $E(X_1 + \cdots + X_6) = E(X_1) + \cdots + E(X_6) = 6E(X_1) = (6)(3)(2/17)(1/16) = 9/68 = 0.1324$.

We also calculate Var$(X_1 + \cdots + X_6) = \text{Cov}(X_1, X_2) = 6E(X_1^2) - E(X_1)E(X_1)) + 30E(X_1X_2) - E(X_1)E(X_2) = 6((3)(2/17)(1/16) - (3)(2/17)/(1/16)(3)(2/17)(1/16)) + 30((3)(2/17)(1/16)(13)/(15) - (3)(2/17)/(1/16)(3)(2/17)(1/16)) = 4329/32368 = 0.1337.$