STAT/MA 41600
Midterm Exam 2 Answers
Friday, November 19, 2018
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1ab. We have $f_X(x) = \frac{d}{dx}F_X(x) = \frac{d}{dx}(\frac{x-3}{4}) = 1/4$ for $3 < x < 7$, and $f_X(x) = 0$ otherwise. So X is a continuous Uniform on $(3, 7)$. Thus $\mathbb{E}(X) = \frac{3+7}{2} = 5$ and $\mathrm{Var}\,(X) = \frac{(7-3)^2}{12} = 4/3$.

2. Let X denote the number of chocolate chip cookies, so X is Binomial with $n = 112$ and $p = 0.4$. So we get $P(X \geq 50) = P(X \geq 49.5) = P\Big(\frac{X-(112)(0.4)}{\sqrt{(112)(0.4)(0.6)}} \geq \frac{49.5-(112)(0.4)}{\sqrt{(112)(0.4)(0.6)}}\Big) \approx P(Z \geq 0.91) = 1 - P(Z \leq 0.91) = 1 - 0.8186 = 0.1814$.

3. We compute $P(Y > X/2) = \int_0^\infty\int_{x/2}^\infty 15e^{-5x-3y}\,dy\,dx = \int_0^\infty -5e^{-5x-3y}|_{y=x/2}^\infty\,dx = \int_0^\infty 5e^{-5x-3x/2}\,dx = \int_0^\infty 5e^{-13x/2}\,dx = -(10/13)e^{-13x/2}|_{x=0}^\infty = 10/13$.

4a. For $x > 0$, we have $f_X(x) = \int_x^\infty 70e^{-3x-7y}\,dy = 10e^{-10x}$; for $x \leq 0$, we have $f_X(x) = 0$.
4b. For $x > 0$, the conditional density is $f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{70e^{-3x-7y}}{10e^{-10x}} = 7e^{7x-7y}$, so we get $f_{Y|X}(y \mid \frac{1}{10}) = 7e^{7(1/10)-7y}$. Therefore, we conclude $P(Y > 1/4 \mid X = 1/10) = \int_{1/4}^\infty f_{Y|X}(y \mid \frac{1}{10})\,dy = e^{-21/20} = 0.3499$.

5a. We compute $\mathrm{Cov}(X_1, X_2) = \mathbb{E}(X_1X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2) = (12/52)(11/51) - (3/13)^2 = -10/2873$.
5b. We have $\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = \mathbb{E}(X_1^2) - (\mathbb{E}(X_1))^2 = \mathbb{E}(X_1) - (\mathbb{E}(X_1))^2 = 3/13 - (3/13)^2 = 30/169$. So we obtain:
$\rho(X_1, X_2) = \mathrm{Cov}(X_1, X_2)/\sqrt{\mathrm{Var}(X_1)\mathrm{Var}(X_2)} = (-10/2873)/\sqrt{(30/169)(30/169)} = -1/51$.

The problems come from:
(1) 2014, PS 31; (2) 2017, PS 37; (3) 2016, PS 25; (4) 2015, PS 27; (5) 2018, PS 39 part 2