STAT/MA 41600
In-Class Problem Set #32 part 2: November 3, 2014
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1a. We have $P(W > 3/5 \mid W > 1/5) = \frac{P(W>3/5\ \&\ W>1/5)}{P(W>1/5)} = \frac{P(W>3/5)}{P(W>1/5)} = \frac{e^{-3}}{e^{-1}} = e^{-2} = 0.1353.$

Alternatively, using the memoryless property of exponential random variables, we have $P(W > 3/5 \mid W > 1/5) = P(W > 2/5) = e^{-(5)(2/5)} = e^{-2}$.

1b. We have $P(W < 1/5 \mid W < 3/5) = \frac{P(W<1/5\ \&\ W<3/5)}{P(W<3/5)} = \frac{P(W<1/5)}{P(W<3/5)} = \frac{1-e^{-1}}{1-e^{-3}} = 0.6652.$

2a. Since X is the minimum of independent exponential random variables, then X is exponential with $\lambda = 3+5 = 8$, so $\mathbb{E}(X) = 1/8$. Alternatively, $\mathbb{E}(X) = \int_0^\infty \int_v^\infty v15e^{-3v-5w}\,dw\,dv + \int_0^\infty \int_w^\infty w15e^{-3v-5w}\,dv\,dw = 3/64 + 5/64 = 8/64 = 1/8$.

2b. We have $\mathbb{E}(Y) = \int_0^\infty \int_v^\infty w15e^{-3v-5w}\,dw\,dv + \int_0^\infty \int_w^\infty v15e^{-3v-5w}\,dv\,dw = 39/320 + 55/192 = 49/120$.

3a. We have $P(X > 3/4 \mid X > 1/4) = \frac{P(X>3/4\ \&\ X>1/4)}{P(X>1/4)} = \frac{P(X>3/4)}{P(X>1/4)} = \frac{e^{-3/2}}{e^{-1/2}} = e^{-1} = 0.3679$. Alternatively, using the memoryless property of exponential random variables, $P(X > 3/4 \mid X > 1/4) = P(X > 1/2) = e^{-(2)(1/2)} = e^{-1}$.

3b. We have $P(X < 1/4 \mid X < 3/4) = \frac{P(X<1/4\ \&\ X<3/4)}{P(X<3/4)} = \frac{P(X<1/4)}{P(X<3/4)} = \frac{1-e^{-1/2}}{1-e^{-3/2}} = 0.5065.$

4a. Since U is the minimum of independent exponential random variables, then U is exponential with $\lambda = 2+7 = 9$, so $\mathbb{E}(U) = 1/9$. Alternatively, $\mathbb{E}(U) = \int_0^\infty \int_x^\infty x14e^{-2x-7y}\,dy\,dx + \int_0^\infty \int_y^\infty y14e^{-2x-7y}\,dx\,dy = 2/81 + 7/81 = 9/81 = 1/9$.

4b. We have $\mathbb{E}(V) = \int_0^\infty \int_x^\infty y14e^{-2x-7y}\,dy\,dx + \int_0^\infty \int_y^\infty x14e^{-2x-7y}\,dx\,dy = 32/567 + 77/162 = 67/126$.

5a. The minimum of independent exponential random variables is an exponential random variable too. The parameter for U is $17 = 3+5+2+7$. Thus $P(U > 1/10) = e^{-17(1/10)} = e^{-17/10} = 0.1827$.

5b. We have $\mathbb{E}(U) = 1/17$.

6a. We have $P(U < 1/5 \mid U > 1/20) = \frac{P(1/20<U<1/5)}{P(U>1/20)} = \frac{e^{-17/20}-e^{-17/5}}{e^{-17/20}} = 1 - e^{-51/20} = 0.9219.$ Alternatively, $P(U < 1/5 \mid U > 1/20) = 1 - P(U > 1/5 \mid U > 1/20)$, but using the memoryless property, this becomes $1 - P(U > 3/20) = 1 - e^{-17(3/20)} = 1 - e^{-51/20}$.

6b. The probability that Y is the smallest of the random variables V, W, X, Y can be computed as: $\int_0^\infty \int_y^\infty \int_y^\infty \int_y^\infty (3e^{-3v})(5e^{-5w})(2e^{-2x})(7e^{-7y})\,dv\,dw\,dx\,dy = \int_0^\infty (e^{-3y})(e^{-5y})(e^{-2y})(7e^{-7y})\,dy = 7\int_0^\infty e^{-17y}\,dy = 7/17$.