

STAT/MA 41600  
 Midterm Exam 1 Answers  
 Friday, October 10, 2014  
 Solutions by Mark Daniel Ward

**1. Method #1:** Let  $B$  denote the event that “black” appears on top, and let  $A$  denote the event that the fair die (with 3 white sides and 3 black sides) was chosen. Then  $P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B | A)P(A)}{P(B | A)P(A) + P(B | A^c)P(A^c)} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (2/3)(1/2)} = 3/7$ .

*Method #2:* There are seven black sides on the dice, and all are equally likely to appear, and three of them belong to the fair die. So the desired probability is  $3/7$ .

**2a.** Let  $X_j$  indicate if the  $j$ th group consists of 1 Stat and 1 Math student, i.e.,  $X_j = 1$  if this happens, and  $X_j = 0$  otherwise. Then  $X = X_1 + \dots + X_{20}$ , so  $\mathbb{E}(X) = \mathbb{E}(X_1 + \dots + X_{20}) = \mathbb{E}(X_1) + \dots + \mathbb{E}(X_{20}) = 20/39 + \dots + 20/39 = 20(20/39) = 400/39 = 10.2564$ .

**2b.** We have  $X^2 = (X_1 + \dots + X_{20})^2$ , so  $\mathbb{E}(X^2) = \mathbb{E}((X_1 + \dots + X_{20})^2)$ , and when we expand, we get 20 terms that each equal  $\mathbb{E}(X_1)$ , and  $20^2 - 20 = 380$  terms that each equal  $\mathbb{E}(X_1 X_2)$ . We already observed  $\mathbb{E}(X_1) = 20/39$ . Also  $\mathbb{E}(X_1 X_2) = P(X_1 X_2 = 1) = (20/39)(19/37)$ . Thus  $\mathbb{E}(X^2) = (20)(20/39) + (380)(20/39)(19/37) = 159200/1443 = 110.3257$ . So  $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 288800/56277 = 5.1318$ .

**3. Method #1:** We have  $P(X \geq y) = \sum_{x=y}^{\infty} \sum_{x=y}^{\infty} p_{X,Y}(x, y) = \sum_{y=1}^{\infty} \sum_{x=y}^{\infty} (\frac{3}{4})^{x-1} (\frac{1}{4})(\frac{1}{2})^y = (\frac{1}{4})(\frac{1}{2}) \sum_{y=1}^{\infty} (\frac{1}{2})^{y-1} \sum_{x=y}^{\infty} (\frac{3}{4})^{x-1}$ . The inner sum is  $\sum_{x=y}^{\infty} (\frac{3}{4})^{x-1} = \frac{(\frac{3}{4})^{y-1}}{1 - \frac{3}{4}}$ . Thus  $P(X \geq Y) = (\frac{1}{4})(\frac{1}{2}) \sum_{y=1}^{\infty} (\frac{1}{2})^{y-1} \frac{(\frac{3}{4})^{y-1}}{1 - \frac{3}{4}} = (\frac{1}{2}) \sum_{y=1}^{\infty} ((\frac{1}{2})(\frac{3}{4}))^{y-1} = (\frac{1}{2}) \sum_{y=1}^{\infty} (\frac{3}{8})^{y-1} = (\frac{1}{2}) / (1 - \frac{3}{8}) = \frac{4}{5}$ .

*Method #2:* We want the probability that the first head occurs earlier, or at the same time, as the first “1.” Of the 8 equally-likely results  $(x, y)$  with  $1 \leq x \leq 4$  and  $y = H$  or  $y = T$ , there are 5 equally-likely results in which  $x = 1$  or  $y = H$  or both, namely:  $(1, T), (1, H), (2, H), (3, H), (4, H)$ . Given that one of these 5 equally-likely results occurs, it is one of the 4 results with H (not T) with probability  $4/5$ .

**4a.** The exact probability is  $\sum_{x=0}^4 \binom{300,000}{x} (\frac{1}{100,000})^x (\frac{99,999}{100,000})^{300,000-x}$ , namely,

$$\begin{aligned} & \binom{300,000}{0} \left(\frac{1}{100,000}\right)^0 \left(\frac{99,999}{100,000}\right)^{300,000} + \binom{300,000}{1} \left(\frac{1}{100,000}\right)^1 \left(\frac{99,999}{100,000}\right)^{299,999} \\ & + \binom{300,000}{2} \left(\frac{1}{100,000}\right)^2 \left(\frac{99,999}{100,000}\right)^{299,998} + \binom{300,000}{3} \left(\frac{1}{100,000}\right)^3 \left(\frac{99,999}{100,000}\right)^{299,997} \\ & + \binom{300,000}{4} \left(\frac{1}{100,000}\right)^4 \left(\frac{99,999}{100,000}\right)^{299,996} \end{aligned}$$

**4b.** If  $n = 300,000$  and  $p = 1/100,000$ , then the probability in part **4a** is the probability that a Binomial  $n, p$  random variable is 4 or less. Now let  $Y$  be Poisson with average  $\lambda = np = 3$ . So the Poisson approximation is  $P(X \leq 4) \approx P(Y \leq 4) = \frac{e^{-3}3^0}{0!} + \frac{e^{-3}3^1}{1!} + \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!} + \frac{e^{-3}3^4}{4!}$ .

**4c.** We have  $P(X \leq 4) \approx P(Y \leq 4) = \frac{e^{-3}3^0}{0!} + \frac{e^{-3}3^1}{1!} + \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!} + \frac{e^{-3}3^4}{4!} = 0.8153$ .

Please see some notes on the reverse side of this sheet.

**Notes.**

Question 1 was the same as Problem Set #5, question 6, using dice instead of coins.

Question 2 was the same as Problem Set #20/#22, question 3, with some numbers changed and with the additional question about the variance. We had extensive practice with these kinds of variance questions, especially throughout Problem Set #12.

Question 3 was the same as Problem Set #16, question 6, with some numbers changed.

Question 4 was the same as in the (take-home) Practice Problems #19, question 5, with some numbers changed. We had practice with these kinds of Poisson approximations to Binomial random variables in Problem Set #18, especially questions 4 and 5.